τ-tilting finite simply connected algebras

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Outline

Auslander-Reiten translation

Throughout, let Λ be a finite dimensional basic algebra over an algebraically closed field K. For a Λ-module M with a minimal projective presentation

$$P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0,$$

we have

$$\mathrm{Tr}M := \text{coker } \mathrm{Hom}_\Lambda(d_1, \Lambda).$$

The Auslander-Reiten translation is defined by

$$\tau M := D\mathrm{Tr}M,$$

where $D = \mathrm{Hom}_K(-, K)$.

Introduction

In 2014, Adachi-Iyama-Reiten introduced support τ-tilting modules for any Λ and constructed the (left or right) mutation of them, which has the following nice properties:

- Mutation (left or right) is always possible.
- There is a partial order on the set of (isomorphism classes of) basic support τ-tilting modules such that its Hasse quiver realizes the left mutation.

This is considered as a generalization of the classical tilting theory via mutations.

We call Λ τ-tilting finite if there are only finitely many (iso. classes of) basic support τ-tilting Λ-modules.

Motivation

Note that representation-finite algebras are τ-tilting finite and the converse is not true in general. For example, let

$$\Lambda_n = K(\bullet \xrightarrow{a} \bullet \circlearrowleft b)/ < b^n, ab^2 >,\ n \geqslant 2,$$

then Λ_n is τ-tilting finite. But Λ_n is

- representation-finite if $n = 2, 3, 4, 5$.
- tame if $n = 6$.
- wild if $n \geqslant 7$.

Therefore, we want to know which kind of algebras meets the following conditions:

$$\tau\text{-}\textbf{tilting finite} \Leftrightarrow \textbf{representation-finite.}$$

Besides, many scholars have studied the support τ-tilting modules of various algebras. For example,

- **Gentle algebras** (Plamondon, 2018).
- **Tilted and cluster-tilted algebras** (Zito, 2019).
- Algebras with radical square zero (Adachi, 2016).
- Brauer graph algebras (Adachi-Aihara-Chan, 2018).
- Preprojective algebras of Dynkin type (Aihara-Mizuno, 2016, Mizuno, 2014).

τ-tilting theory

We denote by $|M|$ the number of (iso. classes of) indecomposable direct summands of M.

Definition 1.1 (Adachi-Iyama-Reiten, 2014)

Let M be a right Λ-module and $P \in \operatorname{proj} \Lambda$.

(1) M is called τ-tilting if $\operatorname{Hom}_\Lambda(M, \tau M) = 0$ and $|M| = |\Lambda|$.

(2) M is called support τ-tilting if M is a τ-tilting $(\Lambda / \langle e \rangle)$-module, where e is an idempotent of Λ.

We denote by $\mathrm{s}\tau\text{-tilt}\ \Lambda$ the set of (iso. classes of) basic support τ-tilting Λ-modules.

Example

Let $\Lambda = K(\ 1 \underset{b}{\overset{a}{\rightleftarrows}} 2\ )/ < ab, ba >$ and we denote

$$S_1 = 1, S_2 = 2, P_1 = {}^{1}_{2}, P_2 = {}^{2}_{1}.$$

Then,

$$\tau(S_1) = S_2, \tau(S_2) = S_1, \tau(P_1) = 0, \tau(P_2) = 0.$$

Thus, for example,

- $P_1 \oplus P_2, P_1 \oplus S_1$ and $S_2 \oplus P_2$ are τ-tilting modules.
- S_1 and S_2 are support τ-tilting modules.

Mutation

We denote by add(M) (respectively, Fac(M)) the full subcategory whose objects are direct summands (respectively, factor modules) of finite direct sums of copies of M.

Definition 1.2 (Adachi-Iyama-Reiten, 2014)

Let $T = M \oplus N$ be a basic τ-tilting module, where $M \notin \mathrm{Fac}(N)$ is an indecomposable summand. We take an exact sequence with a minimal left add(N)-approximation π:

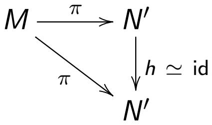
$$M \xrightarrow{\pi} N' \longrightarrow U \longrightarrow 0,$$

we call $\mu_M^-(T) := U \oplus N$ the left mutation of T with respect to M.

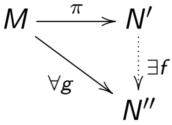
Remark

π is called a minimal left add(N)-approximation if $N' \in \mathrm{add}(N)$ and it satisfies the following conditions:

(i) every $h : N' \to N'$ that satisfies $h \circ \pi = \pi$ is an automorphism.



(ii) for any $N'' \in \mathrm{add}(N)$ and $g : M \to N''$, there exists $f : N' \to N''$ such that $f \circ \pi = g$.



Example

Let $\Lambda = K(1 \underset{b}{\overset{a}{\rightleftarrows}} 2)/ < ab, ba >$, then $P_1 \oplus P_2$ is a τ-tilting module. We consider the left mutation with respect to P_2,

$$P_2 \xrightarrow{\pi} P_1 \longrightarrow \text{coker}\pi \longrightarrow 0,$$

where $\pi : \begin{smallmatrix} e_2 \\ b \end{smallmatrix} \xrightarrow{a\cdot} \begin{smallmatrix} e_1 \\ a \end{smallmatrix}$ is a minimal left add(P_1)-approximation, then

$$\text{coker}\pi = S_1 \text{ and } \mu^-_{P_2}(\Lambda) = P_1 \oplus S_1.$$

In fact, we have the following mutation quiver of sτ-tilt Λ.

$$\begin{array}{ccccc} P_1 \oplus P_2 & \longrightarrow & P_1 \oplus S_1 & \longrightarrow & S_1 \\ \downarrow & & & & \downarrow \\ S_2 \oplus P_2 & \longrightarrow & S_2 & \longrightarrow & 0 \end{array}$$

Poset structure

Definition 1.3 (Adachi-Iyama-Reiten, 2014)

For $M, N \in \text{s}\tau\text{-tilt}\,\Lambda$, we say $M \geqslant N$ if $\text{Fac}(N) \subseteq \text{Fac}(M)$.

Example

Let Λ be the algebra given before. The Hasse quiver of $\text{s}\tau\text{-tilt}\,\Lambda$ is

$$\begin{array}{ccccc} P_1 \oplus P_2 & \xrightarrow{>} & P_1 \oplus S_1 & \xrightarrow{>} & S_1 \\ \downarrow{\scriptstyle >} & & & & \downarrow{\scriptstyle >} \\ S_2 \oplus P_2 & \xrightarrow{>} & S_2 & \xrightarrow{>} & 0 \end{array}$$

Proposition 1.4 (Adachi-Iyama-Reiten, 2014)

The mutation quiver $\mathcal{Q}(\mathsf{s}\tau\text{-tilt}\ \Lambda)$ and the Hasse quiver $\mathcal{H}(\mathsf{s}\tau\text{-tilt}\ \Lambda)$ coincide.

Proposition 1.5 (Adachi-Iyama-Reiten, 2014)

If the mutation quiver $\mathcal{Q}(\mathsf{s}\tau\text{-tilt}\ \Lambda)$ contains a finite connected component, then it exhausts all support τ-tilting modules.

Reduction theorems

It is well-known that any idempotent truncation of a τ-tilting finite algebra is also τ-tilting finite. Furthermore, we have

Proposition 1.6 (Adachi-Iyama-Reiten, 2014)

There exists a poset isomorphism between $\mathsf{s}\tau\text{-tilt}\ \Lambda$ and $\mathsf{s}\tau\text{-tilt}\ \Lambda^{\mathsf{op}}$.

Proposition 1.7 (Eisele-Janssens-Raedschelders, 2018)

Let I be a two-sided ideal generated by elements which are contained in the center and the radical, then there exists a poset isomorphism between $\mathsf{s}\tau\text{-tilt}\ \Lambda$ and $\mathsf{s}\tau\text{-tilt}\ (\Lambda/I)$.

Minimal representation-infinite algebras

An algebra Λ is called minimal rep.-infinite if Λ is rep.-infinite, but $\Lambda/\Lambda e\Lambda$ is rep.-finite for any non-zero idempotent e of Λ.

We denote by Γ_Λ the Auslander-Reiten quiver of Λ. A connected component C of Γ_Λ is called preprojective if

- there is no oriented cycle in C, and
- any module in C is of form $\tau^{-k}(P)$ for some $k \in \mathbb{N}$ and some indecomposable projective module P.

Proposition 1.8 (Happel-Vossieck, 1983)

A m.r.i. algebra with preprojective component is either a n-Kronecker algebra ($n \geqslant 2$) or a tame concealed algebra, which is of type $\widetilde{\mathbb{A}}_n, \widetilde{\mathbb{D}}_n (n \geqslant 4), \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7$ or $\widetilde{\mathbb{E}}_8$.

Tilted algebras

A tilting Λ-module T provided:

- $|T| = |\Lambda|$;
- gl. dim. $T \leqslant 1$;
- $\mathrm{Ext}^1_\Lambda(T, T) = 0$.

A (concealed) tilted algebra of type Q is the endomorphism algebra of a (preprojective) tilting module over a hereditary algebra KQ.

Lemma 1.9 (Zito, 2019)

Let Λ be a tilted or cluster-tilted algebra, then Λ is τ-tilting finite if and only if Λ is representation-finite.

Simply connected algebras

Let $\Lambda = KQ/I$ be an algebra with a quiver $Q = (Q_0, Q_1, s, t)$ and an admissible ideal I. For each arrow $\alpha \in Q_1$, let α^- be its formal inverse with $s(\alpha^-) = t(\alpha)$ and $t(\alpha^-) = s(\alpha)$. Then, we set

$$Q_1^- = \{\alpha^- \mid \alpha \in Q_1\}.$$

A walk is a formal composition $w = w_1 w_2 \dots w_n$ with $w_i \in Q_1 \cup Q_1^-$ for all $1 \leqslant i \leqslant n$. Then, we set

$$s(w) = s(w_1),\ t(w) = t(w_n)$$

and denote by 1_x the trivial path at vertex x.

For walks w and u with $s(u) = t(w)$, the composition wu is defined in the obvious way.

Let $\sim$ be the smallest equivalence relation on the set of all walks in Q satisfying the following conditions:

- For each $\alpha : x \to y$ in Q_1, we have $\alpha\alpha^- \sim 1_x$ and $\alpha^-\alpha \sim 1_y$.
- For each minimal relation $\sum_{i=1}^{n} \lambda_i \omega_i$ in I, we have $\omega_i \sim \omega_j$ for all $1 \leqslant i, j \leqslant n$.
- If u, v, w and w' are walks and $u \sim v$, then $wuw' \sim wvw'$ whenever these compositions are defined.

We denote by $[w]$ the equivalence class of a walk w.

Let $x \in Q_0$. The set $\Pi_1(Q, I, x)$ of equivalence classes of all walks w with $s(w) = t(w) = x$ is a group via $[u] \cdot [v] = [uv]$, and one can show that it does not depend on the choice of x. Thus, we define the fundamental group of (Q, I) as follows.

$$\Pi_1(Q, I) := \Pi_1(Q, I, x).$$

Recall that $\Lambda = KQ/I$ is called triangular if Q is acyclic.

Definition 2.1 (Assem-Skowroński, 1988)

A connected triangular algebra Λ is simply connected if, for every presentation (Q, I) of Λ, the fundamental group $\Pi_1(Q, I)$ is trivial.

We have the following examples.

(1) All tree algebras are simply connected.

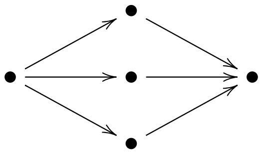
(2) A hereditary algebra is simply connected if and only if its quiver is a tree.

Theorem 2.2 (W, 2019)

Let Λ be a simply connected algebra, then it is τ-tilting finite if and only if Λ is representation-finite.

A full subquiver Q' of Q is a convex subquiver if any path in Q with source and target in Q' lies entirely in Q'. Let $I' := KQ' \cap I$, then KQ'/I' is called a convex subalgebra of KQ/I.

An algebra is called critical if it is rep.-infinite, but any proper convex subalgebra is rep.-finite. Note that the path algebra KQ with the following quiver Q, is critical but not m.r.i..



A grading of a tree T is a function $g : T_{vertex} \to \mathbb{N}$ satisfying

- $g^{-1}(0) \neq \varnothing$.
- $g(x) - g(y) \in 1 + 2\mathbb{Z}$, whenever x and y are neighbours in T.

A graded tree is a pair (T, g) formed by a tree T and a grading g.

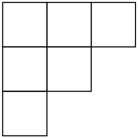
Sketch of the proof:

By [Bongartz, 1984], Λ is rep.-finite if and only if it does not contain a critical convex subalgebra, which arises from a graded tree. On the other hand, such a critical algebra is a m.r.i. algebra with preprojective component.

Staircase algebras

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a (non-increasing) partition of a positive integer n. It is well-known that we can visualize λ by the corresponding Young diagram $Y(\lambda)$. For example,

$$\lambda = (3, 2, 1) \Leftrightarrow Y(\lambda) =$$

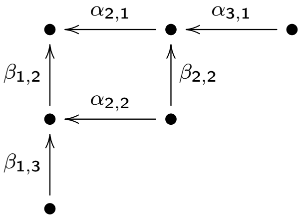


For any partition $\lambda \vdash n$, let

- Q_λ is a quiver such that the vertices are given by the boxes in $Y(\lambda)$, the arrows are given by drawing arrows from right to left and from bottom to top.
- I_λ is a two-sided ideal generated by all commutativity relations for all squares appearing in Q_λ.

Then the algebra $\mathcal{A}(\lambda) := KQ_\lambda / I_\lambda$ is called a staircase algebra.

If $\lambda = (3, 2, 1)$, then Q_λ is given by



Then, the corresponding staircase algebra $\mathcal{A}(\lambda)$ is defined by

$$\mathcal{A}(\lambda) := KQ_\lambda / < \beta_{2,2}\alpha_{2,1} - \alpha_{2,2}\beta_{1,2} >.$$

Proposition 3.1 (Boos, 2017)

For any $\lambda \vdash n$, $\mathcal{A}(\lambda)$ is triangular and simply connected.

Proposition 3.2 (Boos, 2017)

A staircase algebra $\mathcal{A}(\lambda)$ with $\lambda \vdash n$ is

(1) representation-finite if and only if one of the following holds:
   - $\lambda \in \{(n), (n-k, 1^k), (n-2, 2), (2^2, 1^{n-4})\}$ for $k \leqslant n$.
   - $n \leqslant 8$ and $\lambda \notin \{(4,3,1), (3^2, 2), (3, 2^2, 1), (4, 2, 1^2)\}$.

(2) tame concealed if and only if λ comes up in the following list:

$$(6,3), (6,2,1), (5,2^2), (4,3,1), (4,2,1^2),$$
$$(3,2^2,1), (3^2,1^3), (2^3,1^3), (3,2,1^4).$$

(3) tame, but not tame concealed if and only if λ comes up in the following list:

$$(5^2), (5,4), (4^2,1), (3^3), (3^2,2), (3,2^3), (2^5), (2^4,1).$$

Otherwise, $\mathcal{A}(\lambda)$ is wild.

Corollary 3.3 (W, 2019)

A staircase algebra $\mathcal{A}(\lambda)$ with $\lambda \vdash n$ is τ-tilting finite if and only if one of the following holds:

- $\lambda \in \{(n), (n-k, 1^k), (n-2, 2), (2^2, 1^{n-4})\}$ for $k \leqslant n$.
- $n \leqslant 8$ and $\lambda \notin \{(4,3,1), (3^2, 2), (3, 2^2, 1), (4, 2, 1^2)\}$.

Question 3.4

We have known that if $\lambda = (n)$ or $(n-k, 1^k)$, the number of support τ-tilting $\mathcal{A}(\lambda)$-modules is

$$\tfrac{1}{n+2}\tbinom{2n+2}{n+1}.$$

How about others?

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Thank you very much for your attention !