On Schurian finiteness of Schur algebras

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Outline

Introduction

We work with finite-dimensional algebras over an algebraically closed field K. ($\mathbb{C}$ is an algebraically closed field, but $\mathbb{R}$ is not.)

e.g.,

$$A = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \mid a_{ij} \in K \right\}$$

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$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mapsto e_1 \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mapsto e_2 \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mapsto e_3 \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \alpha \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \beta \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \alpha\beta$$

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$$Q_A: \; 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

Then, $A \simeq KQ_A$.

Set $B = A/I$, where I is the ideal

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Quiver Presentation

Any (basic, connected) algebra A over K is isomorphic to a **bound quiver algebra** KQ/I.

Representation of quivers

e.g., $\circ \xrightarrow{\alpha} \circ \xrightarrow{\beta} \circ$. A representation:

$$V_1 \xrightarrow{f_\alpha} V_2 \xrightarrow{f_\beta} V_3$$

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Indecomposable rep's:

$$K \xrightarrow{0} 0 \xrightarrow{0} 0 \qquad K \xrightarrow{1} K \xrightarrow{0} 0$$

$$0 \xrightarrow{0} K \xrightarrow{0} 0 \qquad 0 \xrightarrow{0} K \xrightarrow{1} K$$

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Indecomposable rep's with (some) morphisms:

$$\begin{array}{ccccc} K & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\ 0 & \xrightarrow{0} & K & \xrightarrow{0} & 0 \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\ 0 & \xrightarrow{0} & 0 & \xrightarrow{0} & K \end{array} \qquad \begin{array}{ccccc} K & \xrightarrow{1} & K & \xrightarrow{0} & 0 \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\ 0 & \xrightarrow{0} & K & \xrightarrow{1} & K \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\ K & \xrightarrow{1} & K & \xrightarrow{1} & K \end{array}$$

Representation of quivers

Goal of Representation Theory

Classify all indecomposable rep's of a given quiver Q and all morphisms between them, up to isomorphism.

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$$K^2 \xrightarrow{(1,1)} K \xrightarrow{1} K \simeq \begin{matrix} K \xrightarrow{0} 0 \xrightarrow{0} 0 \\ \oplus \\ K \xrightarrow{1} K \xrightarrow{1} K \end{matrix}$$

Representation of quivers

e.g., $\circ \rightrightarrows \circ$. Indecomposable rep's:

dimension 2: $K \underset{0}{\overset{1}{\rightrightarrows}} K$ $\quad$ $K \underset{\lambda}{\overset{1}{\rightrightarrows}} K$

dimension 3: $K^2 \underset{(0,1)}{\overset{(1,0)}{\rightrightarrows}} K$ $\quad$ $K \underset{(0,1)^t}{\overset{(1,0)^t}{\rightrightarrows}} K^2$

dimension 4: $K^2 \underset{J_2(0)}{\overset{I_2}{\rightrightarrows}} K^2$ $\quad$ $K^2 \underset{J_2(\lambda)}{\overset{I_2}{\rightrightarrows}} K^2$

$\vdots$

$$K^{n+1} \underset{[O,I_n]}{\overset{[I_n,O]}{\rightrightarrows}} K^n \qquad K^n \underset{J_n(\lambda)}{\overset{I_n}{\rightrightarrows}} K^n$$

Representation type of algebras

Theorem (Drozd 1977)

The representation type of any algebra (over K) is exactly one of finite, tame and wild.

An algebra A is said to be

- **finite** if the set of indecomposable rep's is finite.
- **tame** if it is not finite, but all indecomposable rep's are organized in a one-parameter family in each dimension.

Otherwise, A is called wild.

An example of wild algebras

e.g., $\circ \; \Rrightarrow \; \circ$. Indecomposable rep's:

dimension 3: $$K^2 \overset{(1,0)}{\underset{(0,1)}{\overset{a}{\Rrightarrow}}} K \qquad a = (\lambda, \mu)$$

An example of wild algebras

e.g., $\circ \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} \circ$. Indecomposable rep's:

$$\text{dimension 3:} \qquad K^2 \overset{(1,0)}{\underset{(0,1)}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xrightarrow{a}}}}} K \qquad a = (\lambda, \mu)$$

Impossible! to give a complete classification of indecomposable rep's for a wild algebra.

An example of wild algebras

e.g., $\circ \begin{matrix} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{matrix} \circ$. Indecomposable rep's:

dimension 3: $$K^2 \; \begin{matrix} \xrightarrow{(1,0)} \\ \xrightarrow{a} \\ \xrightarrow[(0,1)]{} \end{matrix} \; K \qquad a = (\lambda, \mu)$$

Impossible! to give a complete classification of indecomposable rep's for a wild algebra.

A natural idea

To capture some '**finite**' properties in wild cases.

Schurian finiteness

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- (2) $\Rightarrow$ (1) is obvious.
- (1) $\Rightarrow$ (2) is not verified; no counterexample.

Wild, but Schurian-finite

e.g., set $\Lambda_n = KQ/I_n$ with

$$Q: \circ \xrightarrow{\alpha} \circ \circlearrowright \beta \quad \text{and } I_n :< \beta^n, \alpha\beta^2 >, \ n \geqslant 2,$$

then Λ_n is Schurian-finite.

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But the representation type of Λ_n is

- finite if $n \leqslant 5$;
- tame if $n = 6$;
- wild if $n \geqslant 7$.

Schur algebras

Background

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(1) Hecke alg. in type ABD;

(2) Cyclotomic quiver Hecke alg. of level 1 in affine type ACD; of level 2 in affine type A; of level k in affine type A;

(3) Schur/q-Schur/Borel-Schur/infinitesimal-Schur alg.;

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(2) Cyclotomic quiver Hecke alg. of level 1 in affine type ACD; of level 2 in affine type A; (Ariki-Park 2014, 2015, Ariki 2017); of level k in affine type A; (Ariki-Song-W., writing paper...)

(3) Schur/q-Schur/Borel-Schur/infinitesimal-Schur alg.; (Erdmann 1993, Doty-Erdmann-Martin 1999, Erdmann-Nakano 2001, etc)

(4) block alg. of category $\mathcal{O}$; (Futorny-Nakano-Pollack 1999, Boe-Nakano 2005, etc)

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Now, the Schurian finiteness is determined for

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(2) (most) block alg. of Hecke alg. in type A; (Ariki-Speyer 2021, Lyle-Speyer 2022)

In particular, partial results are also obtained for

(3) q-Schur alg.; (tame cases are deduced from (1))

(4) Borel-Schur alg.; (only 3 small cases remaining)

(5) block alg. of Hecke alg. in type B; (Ariki-Speyer-W., working on...)

Schur algebras

- n, r: positive integers;
- $\mathbb{F}$: algebraically closed field of characteristic $p > 0$;
- V: vector space over $\mathbb{F}$ with basis $\{v_1, v_2, \ldots, v_n\}$;
- G_r: the symmetric group on r symbols;

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The r-fold tensor product $V^{\otimes r} := V \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} V$ has a $\mathbb{F}$-basis

$$\{v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r} \mid 1 \leqslant i_j \leqslant n \text{ for all } 1 \leqslant j \leqslant r\},$$

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with a G_r-action on right by

$$(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r}) \cdot \sigma = v_{i_{\sigma(1)}} \otimes v_{i_{\sigma(2)}} \otimes \cdots \otimes v_{i_{\sigma(r)}},$$

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for any $\sigma \in G_r$. The **Schur algebra**:

$$S(n,r) := \mathrm{End}_{\mathbb{F}G_r}(V^{\otimes r}).$$

Schurian finiteness of Schur algebras

(1) The Schurian finiteness of $S(n,r)$ over $p=2$

$n \backslash r$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	···
2	F	F	F	T	F	W	F	W	T	W	T	W	W	W	W	W	W	W	W	W	W	···

$n \backslash r$	1	2	3	4	5	6	7	8	9	10	11	12	13	···
3	F	F	F	W	W	W	W	W	W	W	W	W	W	···
4	F	F	F	W	W	W	W	W	W	W	W	W	W	···
5	F	F	F	W	W	W	W	W	W	W	W	W	W	···
6	F	F	F	W	W	W	W	W	W	W	W	W	W	···
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

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Blue: Schurian finite **Orange**: Schurian infinite

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2	F	F	F	T	F	W	F	W	T	W	T	W	W	W	W	W	W	W	W	W	W	⋯

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3	F	F	F	W	W	W	W	W	W	W	W	W	W	⋯
4	F	F	F	W	W	W	W	W	W	W	W	W	W	⋯
5	F	F	F	W	W	W	W	W	W	W	W	W	W	⋯
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⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

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2	F (blue)	F (blue)	F (blue)	T (blue)	F (blue)	W (blue)	F (blue)	W (blue)	T (blue)	W (orange)	T (blue)	W (orange)	W (blue)	W (orange)	W (blue)	W (orange)	W (blue)	W (orange)	W (blue)	W (orange)	W (orange)	$\cdots$ (orange)

n \ r	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
3	F (blue)	F (blue)	F (blue)	W (blue)	W (blue)	W (orange)	W (orange)	W (orange)	W (orange)	W (orange)	W (orange)	W (orange)	W (orange)	$\cdots$ (orange)
4	F (blue)	F (blue)	F (blue)	W (orange)	W (blue)	W (orange)	W (orange)	W (orange)	W (orange)	W (orange)	W (orange)	W (orange)	W (orange)	$\cdots$ (orange)
5	F (blue)	F (blue)	F (blue)	W (orange)	W (orange)	W (orange)	W (orange)	W (orange)	W (orange)	W (orange)	W (orange)	W (orange)	W (orange)	$\cdots$ (orange)
6	F (blue)	F (blue)	F (blue)	W (orange)	W (orange)	W (orange)	W (orange)	W (orange)	W (orange)	W (orange)	W (orange)	W (orange)	W (orange)	$\cdots$ (orange)
$\vdots$	$\vdots$ (blue)	$\vdots$ (blue)	$\vdots$ (blue)	$\vdots$ (orange)	$\vdots$ (orange)	$\vdots$ (orange)	$\vdots$ (orange)	$\vdots$ (orange)	$\vdots$ (orange)	$\vdots$ (orange)	$\vdots$ (orange)	$\vdots$ (orange)	$\vdots$ (orange)	$\ddots$ (orange)

- The representation type of Schur algebras is completely determined by [Erdmann, 1993], [Xi, 1993], [Doty-Nakano, 1998] and [Doty-Erdmann-Martin-Nakano, 1999].

(2) The Schurian finiteness of $S(n,r)$ over $p=3$

n \ r	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
2	F	F	F	F	F	F	F	F	T	T	T	W	W	$\cdots$
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5	F	F	F	F	F	W	W	W	W	W	W	W	W	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

(3) The Schurian finiteness of $S(n,r)$ over $p\geqslant 5$

n \ r	$1\sim p-1$	$p\sim 2p-1$	$2p\sim p^2-1$	$p^2\sim p^2+p-1$	$p^2+p\sim\infty$
2	F	F	F	W	W
3	F	F	W	W	W
4	F	F	W	W	W
5	F	F	W	W	W
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Sketch of the proof

(1) Show that if $S(n, r)$ is Schurian infinite, then

- $S(n, n+r)$ is Schurian infinite, and
- $S(N, r)$ for any $N > n$ is Schurian infinite.

Thus, reduce the problem to small n and r.

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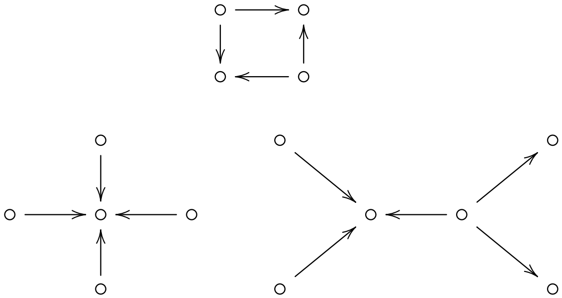
(2) Find the quiver presentation of $S(n, r)$ with small n and r, as well as to **check the Schurian finiteness**.

(3) Gradually enlarge n and r, and repeat (2) until finding a complete classification.

Key observations on Schur algebras

(1) $S(n, r) = KQ/I$ is Schurian infinite if Q contains one of the following quivers as a subquiver.



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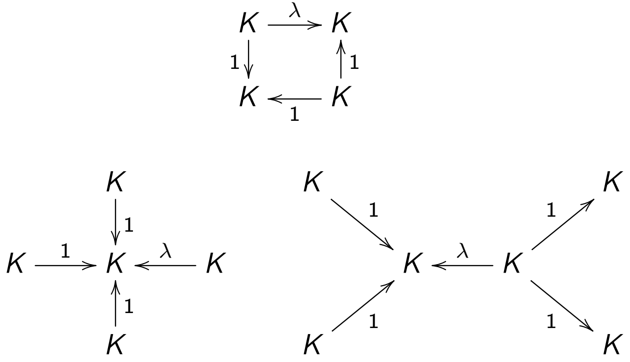

τ-tilting theory

[Adachi-Iyama-Reiten, 2014]

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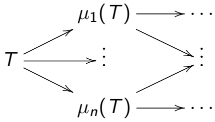
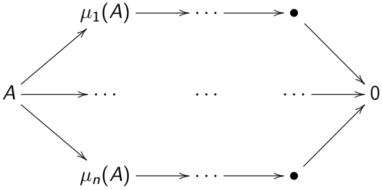
and $\mu_j(T)$ is again a support τ-tilting module. Then, $\mu_j(T)$ is called the mutation of T with respect to T_j.

Mutation graph:



τ-tilting finiteness

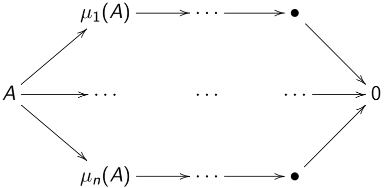
- A is τ-tilting finite if the following mutation graph is finite.



τ-tilting finiteness

Schurian finiteness $\Leftrightarrow$ τ-tilting finiteness

- A is τ-tilting finite if the following mutation graph is finite.



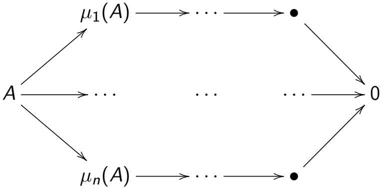
τ-tilting finiteness

Schurian finiteness $\Leftrightarrow$ τ-tilting finiteness

(Schurian rep's $\xleftrightarrow{1:1}$ ind. τ-rigid modules)

[Demonet-Iyama-Jasso, 2018]

- A is τ-tilting finite if the following mutation graph is finite.



Key observations on Schur algebras

(2) For Schur algebra $S(n, r)$, the mutation graph admits a symmetry. For example,

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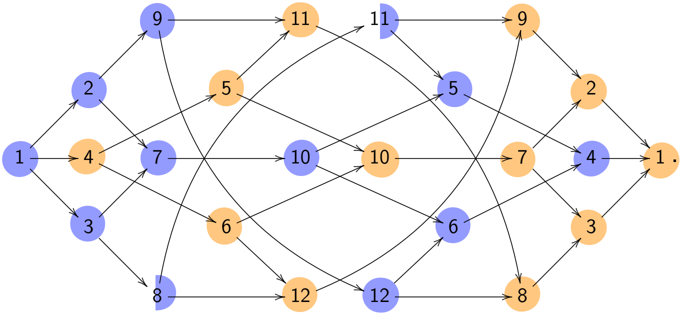

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Thank you! Any questions?

Tools $\begin{cases} \text{Quiver presentation;} \\ \text{Representation type: finite, tame, wild;} \\ \text{Schurian finiteness.} \end{cases}$

Objects $\begin{cases} \text{Hecke algebras;} \\ \text{Schur algebras;} \\ \text{KLR algebras.} \end{cases}$

How? $\begin{cases} \tau\text{-tilting finiteness;} \\ \tau\text{-tilting infinite quiver;} \\ \text{Mutation and mutation graph;} \\ \text{Symmetry on mutation graph.} \end{cases}$