On 2-term silting finiteness of Borel-Schur algebras

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Outline

- Λ: a finite-dimensional algebra over $K = \overline{K}$
- $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\ \Lambda)$: the bounded derived category of mod Λ
- $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\ \Lambda)$: the perfect derived category of mod Λ

Theorem (Rickard, 1989)

An algebra Γ is derived equivalent to Λ, i.e.,

$$\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\ \Gamma) \xrightarrow{\sim} \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\ \Lambda),$$

if and only if there is a tilting complex T in $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\ \Lambda)$ such that

$$\Gamma \simeq \mathrm{End}_{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\ \Lambda)}(T).$$

Tilting mutation: $T = T_1 \oplus \cdots \oplus T_j \oplus \cdots \oplus T_n \quad \in \operatorname{tilt} \Lambda$

$$\Rightarrow \quad \mu_j(T) = T_1 \oplus \ldots \oplus T_j^* \oplus \ldots \oplus T_n \quad \in \operatorname{tilt} \Lambda$$

with $T_j^* \not\simeq T_j$. (See [Riedtmann-Schofield, 1991])

Problem: Tilting mutation is not always possible.

Progress: the following mutations are always possible.

- silting mutation of silting complexes [Aihara-Iyama, 2012].
- mutation of support τ-tilting modules [Adachi-Iyama-Reiten, 2014].
- mutation of certain cluster tilting objects [Buan-Marsh-Reineke-Reiten-Todorov, 2006].

Advantages of silting theory: e.g.,

- the set of silting complexes admits a partial order, such that its Hasse quiver realizes the silting mutation,
- the set of 2-term silting complexes is in bijection with the set of support τ-tilting modules.

In this talk, we are going to explain

- a certain symmetry of the Hasse quiver,
- sign decomposition in silting theory,

as well as their application on Borel-Schur algebras.

Silting Theory

Set $\mathcal{K}_\Lambda := \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\ \Lambda)$. For any $T \in \mathcal{K}_\Lambda$, we fix

- thick T: the smallest thick subcategory of $\mathcal{K}_\Lambda$ containing T,
- add(T): the full subcategory of $\mathcal{K}_\Lambda$, whose objects are direct summands of finite direct sums of copies of T.

Definition 1.1 (Aihara-Iyama, 2012)

A complex $T \in \mathcal{K}_\Lambda$ is said to be

(1) presilting if $\mathrm{Hom}_{\mathcal{K}_\Lambda}(T, T[i]) = 0$, for any $i > 0$.

(2) silting if T is presilting and thick $T = \mathcal{K}_\Lambda$.

(3) tilting if T is silting and $\mathrm{Hom}_{\mathcal{K}_\Lambda}(T, T[i]) = 0$, for any $i < 0$.

e.g., Λ is always a tilting complex in $\mathcal{K}_\Lambda$.

A partial order on silt Λ

silt Λ: the set of iso. classes of silting complexes in $\mathcal{K}_\Lambda$.

Definition 1.2 (Aihara-Iyama, 2012)

For any $T, S \in$ silt Λ, we say $T \geq S$ if

$$\mathrm{Hom}_{\mathcal{K}_\Lambda}(T, S[i]) = 0$$

for any $i > 0$.

Then, $\geq$ gives a partial order on the set silt Λ.

Silting mutation

Theorem-Definition 1.3 (Aihara-Iyama, 2012)

For any $S, T \in \operatorname{silt} \Lambda$, the following conditions are equivalent.

(1) S is a irreducible left mutation of T.

(2) T is a irreducible right mutation of S.

(3) $T > S$ and there is no $X \in \operatorname{silt} \Lambda$ such that $T > X > S$.

Let $T = T_1 \oplus \cdots \oplus T_j \oplus \cdots \oplus T_n \in \operatorname{silt}\Lambda$ with a direct summand T_j. Take a minimal left $\operatorname{add}(T/T_j)$-approximation π and a triangle

$$T_j \xrightarrow{\pi} Z \longrightarrow \operatorname{cone}(\pi) \longrightarrow T_j[1],$$

where $\operatorname{cone}(\pi)$ is the mapping cone of π. Then,

$$\mu_j^-(T) := T_1 \oplus \cdots \oplus \operatorname{cone}(\pi) \oplus \cdots \oplus T_n$$

is again a silting complex in $\mathcal{K}_\Lambda$.

We call $\mu_j^-(T)$ the left silting mutation of T with respect to T_j. Dually, we may define $\mu_j^+(T)$.

Example 1

Let $\Lambda := K(\ 1 \xrightarrow{\alpha_1} 2 \xleftarrow{\alpha_2} 3 \xrightarrow{\alpha_3} 4\ )$. Then,

$$\mu^-_{P_2\oplus P_4}(\Lambda) = \begin{bmatrix} P_2 \xrightarrow{(\alpha_1,\alpha_2)^t} P_1 \oplus P_3 \\ \oplus \\ P_4 \xrightarrow{\alpha_3} P_3 \\ \oplus \\ 0 \longrightarrow P_1 \oplus P_3 \end{bmatrix}, \qquad \mu^+_{(P_1\oplus P_3)[1]}(\Lambda[1]) = \begin{bmatrix} P_2 \oplus P_4 \xrightarrow{(\alpha_2,\alpha_3)} P_3 \\ \oplus \\ P_2 \xrightarrow{\alpha_1} P_1 \\ \oplus \\ P_2 \oplus P_4 \longrightarrow 0 \end{bmatrix}.$$

Remark: In general,

$$\mu^-_{P_i\oplus P_j}(\Lambda) \not\simeq \mu^-_{P_i}(\mu^-_{P_j}(\Lambda)) \not\simeq \mu^-_{P_j}(\mu^-_{P_i}(\Lambda))$$

2-term silting complex

A complex in $\mathcal{K}_\Lambda$ is called 2-term if it is homotopy equivalent to a complex T of the form

$$\cdots \longrightarrow 0 \longrightarrow T^{-1} \xrightarrow{d_T^{-1}} T^0 \longrightarrow 0 \longrightarrow \cdots .$$

We denote by 2-silt Λ the subset of 2-term complexes in silt Λ.

Proposition 1.4 (Adachi-Iyama-Reiten, 2014)

Let U be a 2-term presilting complex in $\mathcal{K}_\Lambda$ with $|U| = |\Lambda| - 1$. Then, U is a direct summand of exactly two 2-term silting complexes in 2-silt Λ.

$\Rightarrow \mu_j^-(\mu_j^-(T))$ is out of 2-silt Λ.

Proposition 1.5 (Aihara-Iyama, 2012)
Let $T = (T^{-1} \to T^0) \in \text{2-silt}\,\Lambda$. Then,

$$\text{add}\,\Lambda = \text{add}\,(T^0 \oplus T^{-1}), \quad \text{add}\,T^0 \cap \text{add}\,T^{-1} = 0.$$

$\Rightarrow$ A 2-term silting complex T must be of the form

$$\left(\bigoplus_{i \in I} P_i^{\oplus a_i} \longrightarrow \bigoplus_{j \in J} P_j^{\oplus a_j} \right), \quad P_k \in \text{proj}\,\Lambda,$$

with $I \cup J = \{1, 2, \ldots, n\}$ and $I \cap J = \emptyset$.

g-vector

If a 2-term complex T in $\mathcal{K}_\Lambda$ is written as

$$\left(\bigoplus_{i=1}^{n} P_i^{\oplus b_i} \longrightarrow \bigoplus_{i=1}^{n} P_i^{\oplus a_i}\right),$$

the class $[T]$ in the Grothendieck group $K_0(\mathcal{K}_\Lambda)$ can be identified by the so-called g-vector

$$g(T) := (a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n) \in \mathbb{Z}^n.$$

$\Rightarrow$ Each entry of $g(T)$ is either > 0 or < 0, for $T \in$ 2-silt Λ.

Proposition 1.6 (Adachi-Iyama-Reiten, 2014)

Let $T \in$ 2-silt Λ. Then, the map $T \mapsto g(T)$ is an injection.

Symmetry on $\mathcal{H}(\text{2-silt}\ \Lambda)$

Set $(-)^* := \mathrm{Hom}_?(-, \Lambda)$ for $? = \mathcal{K}_\Lambda$ or $\mathcal{K}_{\Lambda^{\mathrm{op}}}$. For $T \in \text{2-silt}\ \Lambda$,

$$T = \left(0 \longrightarrow \bigoplus_{i \in I} P_i^{\oplus a_i} \xrightarrow{d} \bigoplus_{j \in J} P_j^{\oplus a_j} \longrightarrow 0 \right),$$

with $I \cup J = \{1, 2, \ldots, n\}$ and $I \cap J = \emptyset$. Then,

$$T^* = \left(0 \longrightarrow 0 \longrightarrow \bigoplus_{j \in J} (P_j^*)^{\oplus a_j} \xrightarrow{d^*} \bigoplus_{i \in I} (P_i^*)^{\oplus a_i} \right).$$

If there is an algebra isomorphism $\sigma : \Lambda^{\mathrm{op}} \to \Lambda$, then σ induces a permutation on $\{1, 2, \ldots, n\}$ by $\sigma(e_i^*) = e_j$. We then obtain an equivalence $\mathcal{K}_{\Lambda^{\mathrm{op}}} \to \mathcal{K}_\Lambda$, also denoted by σ.

We have

$$\sigma(T^*) = \left(0 \longrightarrow 0 \longrightarrow \bigoplus_{j\in J}(P_{\sigma(j)})^{\oplus a_{\sigma(j)}} \xrightarrow{\sigma(d^*)} \bigoplus_{i\in I}(P_{\sigma(i)})^{\oplus a_{\sigma(i)}} \right),$$

which is again a silting complex in $\mathcal{K}_\Lambda$. Set $S_\sigma := [1] \circ \sigma \circ (-)^*$.

Theorem 1.7 (Aihara-W., 2022)

The functor S_σ induces an anti-automorphism of the poset 2-silt Λ. For any $T \in$ 2-silt Λ with $g(T) = (a_1, a_2, \ldots, a_n)$, we have

$$g(S_\sigma(T)) = -(a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)}).$$

Let $\sigma : \Lambda^{\mathrm{op}} \to \Lambda$ be an algebra isomorphism.

- σ fixes a primitive idempotent e, i.e., $\sigma(P^*) = P$;
- σ fixes no primitive idempotent, see an example later.

We define two subsets of 2-silt Λ by

$$\begin{aligned} \mathcal{T}_P^- &:= \{ T \in \text{2-silt}\,\Lambda \mid \mu_P^-(\Lambda) \geq T \geq \Lambda[1] \} \text{ and} \\ \mathcal{T}_P^+ &:= \{ T \in \text{2-silt}\,\Lambda \mid \Lambda \geq T \geq \mu_{P[1]}^+(\Lambda[1]) \}. \end{aligned}$$

Lemma 1.8

We have $\mathcal{T}_P^- \sqcup \mathcal{T}_P^+ = \text{2-silt}\,\Lambda$.

For any $T \in \mathcal{T}_P^- = \{T \in \text{2-silt}\,\Lambda \mid \mu_P^-(\Lambda) \geq T \geq \Lambda[1]\}$,

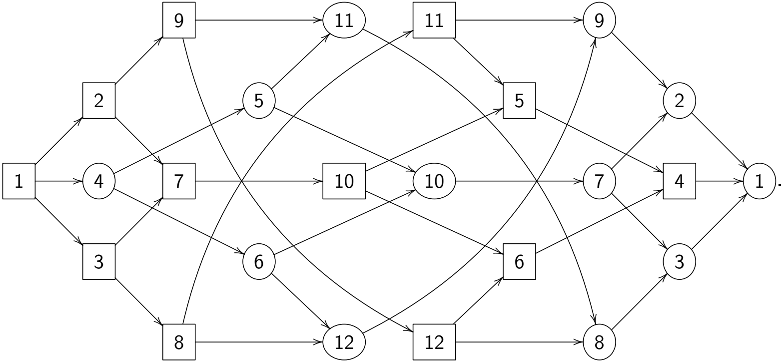
$$\Lambda \geq S_\sigma(T) \geq \mu^+_{\sigma(P^*)[1]}(\Lambda[1]),$$

i.e., $S_\sigma(T) \in \mathcal{T}^+_{\sigma(P^*)}$. Thus,

Theorem 1.9 (Aihara-W., 2022)

If $\sigma(P^*) = P$, then S_σ gives a bijection between $\mathcal{T}_P^-$ and $\mathcal{T}_P^+$.

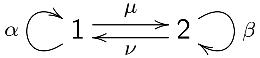
Preprojective algebra of type $\mathbb{A}_3$



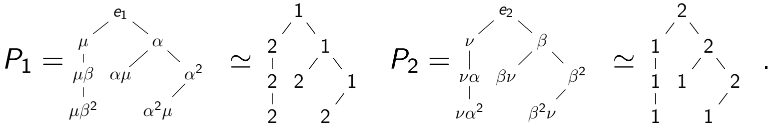
$\mathcal{T}_{P_2}^{-}$ and $\mathcal{T}_{P_2}^{+}$

An example

Let $\Lambda = KQ/I$ be the algebra presented by

$Q:$ 

and $I = \langle \alpha^3, \beta^3, \mu\nu, \nu\mu, \alpha\mu\beta, \beta\nu\alpha, \nu\alpha\mu, \mu\beta\nu, \nu\alpha^2\mu, \mu\beta^2\nu \rangle$. Then,



There exist two algebra isomorphisms $\sigma, \sigma' : \Lambda^{\mathrm{op}} \to \Lambda$ satisfying

$$\sigma(e_1^*) = e_2,\ \sigma(e_2^*) = e_1 \text{ and } \sigma'(e_1^*) = e_1,\ \sigma'(e_2^*) = e_2.$$

By direct calculation, we find the following chain T in $\mathcal{H}(\text{2-silt}\ \Lambda)$,

$$\begin{bmatrix} 0 \longrightarrow P_1 \\ \oplus \\ 0 \longrightarrow P_2 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 \longrightarrow P_1 \\ \oplus \\ P_2 \xrightarrow{f_1} P_1^{\oplus 3} \end{bmatrix} \longrightarrow \begin{bmatrix} P_2 \xrightarrow{f_2} P_1^{\oplus 2} \\ \oplus \\ P_2 \xrightarrow{f_1} P_1^{\oplus 3} \end{bmatrix} \longrightarrow \begin{bmatrix} P_2 \xrightarrow{f_2} P_1^{\oplus 2} \\ \oplus \\ P_2^{\oplus 2} \xrightarrow{f_3} P_1^{\oplus 3} \end{bmatrix} \longrightarrow \begin{bmatrix} P_2 \xrightarrow{\mu} P_1 \\ \oplus \\ P_2^{\oplus 2} \xrightarrow{f_3} P_1^{\oplus 3} \end{bmatrix}$$

with $f_1 = \begin{pmatrix} \mu \\ \mu\beta \\ \mu\beta^2 \end{pmatrix}$, $f_2 = \begin{pmatrix} \mu \\ \mu\beta \end{pmatrix}$, $f_3 = \begin{pmatrix} \mu & 0 \\ -\mu\beta & \mu \\ 0 & \mu\beta \end{pmatrix}$. Then,

$$g(\mathsf{T}): \quad \begin{matrix} (1,0) \\ \oplus \\ (0,1) \end{matrix} \longrightarrow \begin{matrix} (1,0) \\ \oplus \\ (3,-1) \end{matrix} \longrightarrow \begin{matrix} (2,-1) \\ \oplus \\ (3,-1) \end{matrix} \longrightarrow \begin{matrix} (2,-1) \\ \oplus \\ (3,-2) \end{matrix} \longrightarrow \begin{matrix} (1,-1) \\ \oplus \\ (3,-2) \end{matrix}.$$

There are other left mutation chains

$$S_\sigma(\mathsf{T}), \quad S_{\sigma'}(\mathsf{T}), \quad S_{\sigma'}(S_\sigma(\mathsf{T})) = S_\sigma(S_{\sigma'}(\mathsf{T}))$$

in $\mathcal{H}(\text{2-silt}\ \Lambda)$, whose g-vectors are displayed as follows.

$$g(\mathsf{T}): \quad \begin{matrix}(1,0)\\ \oplus\\ (0,1)\end{matrix} \longrightarrow \begin{matrix}(1,0)\\ \oplus\\ (3,-1)\end{matrix} \longrightarrow \begin{matrix}(2,-1)\\ \oplus\\ (3,-1)\end{matrix} \longrightarrow \begin{matrix}(2,-1)\\ \oplus\\ (3,-2)\end{matrix} \longrightarrow \begin{matrix}(1,-1)\\ \oplus\\ (3,-2)\end{matrix}.$$

$g(S_\sigma(\mathsf{T}))$:

$$\begin{matrix}(1,-1)\\ \oplus\\ (2,-3)\end{matrix} \longrightarrow \begin{matrix}(1,-2)\\ \oplus\\ (2,-3)\end{matrix} \longrightarrow \begin{matrix}(1,-2)\\ \oplus\\ (1,-3)\end{matrix} \longrightarrow \begin{matrix}(0,-1)\\ \oplus\\ (1,-3)\end{matrix} \longrightarrow \begin{matrix}(0,-1)\\ \oplus\\ (-1,0)\end{matrix}.$$

$g(S_{\sigma'}(\mathsf{T}))$:

$$\begin{matrix}(-1,1)\\ \oplus\\ (-3,2)\end{matrix} \longrightarrow \begin{matrix}(-2,1)\\ \oplus\\ (-3,2)\end{matrix} \longrightarrow \begin{matrix}(-2,1)\\ \oplus\\ (-3,1)\end{matrix} \longrightarrow \begin{matrix}(-1,0)\\ \oplus\\ (-3,1)\end{matrix} \longrightarrow \begin{matrix}(-1,0)\\ \oplus\\ (0,-1)\end{matrix}.$$

$g(S_{\sigma'}(S_\sigma(\mathsf{T}))) = g(S_\sigma(S_{\sigma'}(\mathsf{T})))$:

$$\begin{matrix}(1,0)\\ \oplus\\ (0,1)\end{matrix} \longrightarrow \begin{matrix}(-1,3)\\ \oplus\\ (0,1)\end{matrix} \longrightarrow \begin{matrix}(-1,3)\\ \oplus\\ (-1,2)\end{matrix} \longrightarrow \begin{matrix}(-2,3)\\ \oplus\\ (-1,2)\end{matrix} \longrightarrow \begin{matrix}(-2,3)\\ \oplus\\ (-1,1)\end{matrix}.$$

The g-vectors for Λ must be given by

$$
\begin{array}{ccccccccccccc}
\substack{(1,0)\\ \oplus \\ (0,1)} & \longrightarrow & \substack{(-1,3)\\ \oplus \\ (0,1)} & \longrightarrow & \substack{(-1,3)\\ \oplus \\ (-1,2)} & \longrightarrow & \substack{(-2,3)\\ \oplus \\ (-1,2)} & \longrightarrow & \substack{(-2,3)\\ \oplus \\ (-1,1)} & \longrightarrow & \substack{(-3,2)\\ \oplus \\ (-1,1)} & \longrightarrow & \substack{(-3,2)\\ \oplus \\ (-2,1)} \\
\downarrow & & & & & & & & & & & & \downarrow \\
\substack{(1,0)\\ \oplus \\ (3,-1)} & & & & & & & & & & & & \substack{(-3,1)\\ \oplus \\ (-2,1)} \\
\downarrow & & & & & & & & & & & & \downarrow \\
\substack{(2,-1)\\ \oplus \\ (3,-1)} & & & & & & & & & & & & \substack{(-3,1)\\ \oplus \\ (-1,0)} \\
\downarrow & & & & & & & & & & & & \downarrow \\
\substack{(2,-1)\\ \oplus \\ (3,-2)} & \longrightarrow & \substack{(1,-1)\\ \oplus \\ (3,-2)} & \longrightarrow & \substack{(1,-1)\\ \oplus \\ (2,-3)} & \longrightarrow & \substack{(1,-2)\\ \oplus \\ (2,-3)} & \longrightarrow & \substack{(1,-2)\\ \oplus \\ (1,-3)} & \longrightarrow & \substack{(0,-1)\\ \oplus \\ (1,-3)} & \longrightarrow & \substack{(-1,0)\\ \oplus \\ (0,-1)}
\end{array}.
$$

Sign Decomposition

[Aoki, 2018], [Aoki-Higashitani-Iyama-Kase-Mizuno, 2022]

Set $g(T) = (x_1, x_2, \ldots, x_n)$ for $T \in 2\text{-silt}\,\Lambda$.
We have either $x_i > 0$ or $x_i < 0$.

Let $\mathsf{s}_n := \{\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)\colon \{1, 2, \ldots, n\} \to \{\pm\}\}$.

We define

$$2\text{-silt}_\epsilon\,\Lambda := \{\, T \in 2\text{-silt}\,\Lambda \mid \epsilon_i x_i > 0,\ 1 \le i \le n \} \,.$$

This gives

$$2\text{-silt}\,\Lambda = \bigsqcup_{\epsilon \in \mathsf{s}_n} 2\text{-silt}_\epsilon\,\Lambda.$$

Structure of $2\text{-silt}_\epsilon \Lambda$

Suppose $I \cup J = \{1, 2, \ldots, n\}$ with

$$I = \{1 \leq i \leq n \mid \epsilon_i = -\} \quad \text{and} \quad J = \{1 \leq j \leq n \mid \epsilon_j = +\}.$$

We define

$$P_I := \oplus_{i \in I} P_i \quad \text{and} \quad P_J := \oplus_{j \in J} P_j.$$

Proposition 2.1 (W., 2023)

We have

$$2\text{-silt}_\epsilon \Lambda = \{ T \in 2\text{-silt}\, \Lambda \mid \mu^-_{P_I}(\Lambda) \geq T \geq \mu^+_{P_J[1]}(\Lambda[1]) \}.$$

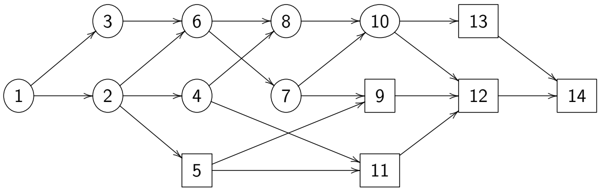
$\Rightarrow$ If the Hasse quiver $\mathcal{H}(2\text{-silt}_\epsilon \Lambda)$ contains a finite connected component $\mathcal{C}$, then $\mathcal{C}$ exhausts all elements of $2\text{-silt}_\epsilon \Lambda$.

Example 1

Let $\Lambda := K(\ 1 \xrightarrow{\alpha_1} 2 \xleftarrow{\alpha_2} 3 \xrightarrow{\alpha_3} 4\ )$. We take $\epsilon = (+,-,+,-)$, i.e., $I = \{2,4\}$ and $J = \{1,3\}$. Then,

$$\mu^-_{P_2 \oplus P_4}(\Lambda) = \left[\begin{array}{c} P_2 \xrightarrow{(\alpha_1,\alpha_2)^t} P_1 \oplus P_3 \\ \oplus \\ P_4 \xrightarrow{\alpha_3} P_3 \\ \oplus \\ 0 \longrightarrow P_1 \oplus P_3 \end{array}\right], \qquad \mu^+_{(P_1 \oplus P_3)[1]}(\Lambda[1]) = \left[\begin{array}{c} P_2 \oplus P_4 \xrightarrow{(\alpha_2,\alpha_3)} P_3 \\ \oplus \\ P_2 \xrightarrow{\alpha_1} P_1 \\ \oplus \\ P_2 \oplus P_4 \longrightarrow 0 \end{array}\right].$$

The Hasse quiver $\mathcal{H}(2\text{-silt}_\epsilon \Lambda)$ has a connected component:



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A simpler algebra

For each $\epsilon \in \mathsf{s}_n$, set

$$e_{\epsilon,+} := \sum_{\epsilon_i=+} e_i \quad \text{and} \quad e_{\epsilon,-} := \sum_{\epsilon_i=-} e_i .$$

Definition 2.2

$$\Lambda_\epsilon := \begin{pmatrix} e_{\epsilon,+}\Lambda e_{\epsilon,+}/J_{\epsilon,+} & e_{\epsilon,+}\Lambda e_{\epsilon,-} \\ 0 & e_{\epsilon,-}\Lambda e_{\epsilon,-}/J_{\epsilon,-} \end{pmatrix}.$$

Here, $J_{\epsilon,+}$ is the two-sided ideal of $e_{\epsilon,+}\Lambda e_{\epsilon,+}$ generated by all $x \in \mathrm{rad}(e_{\epsilon,+}\Lambda e_{\epsilon,+})$ satisfying $xy = 0$ for any $y \in e_{\epsilon,+}\Lambda e_{\epsilon,-}$.

Example 2

Let $\Lambda := K(1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} 4)$.

- $\epsilon = (+,+,+,+)$.
- $\epsilon = (+,-,+,+)$.
- $\epsilon = (+,-,+,-)$.

Hom	1^+	2^-	3^+	4^-
1^+	e_1	0	0	0
2^-	α_1	e_2	0	0
3^+	$\alpha_1\alpha_2$	α_2	e_3	0
4^-	$\alpha_1\alpha_2\alpha_3$	$\alpha_2\alpha_3$	α_3	e_4

$\Rightarrow$

Hom	1^+	2^-	3^+	4^-
1^+	e_1	0	0	0
2^-	α_1	e_2	0	0
3^+	β_1	0	e_3	0
4^-	$\alpha_1\beta_2 = \beta_1\alpha_3$	β_2	α_3	e_4

Some useful observations:

- If a maximal path $w \in e_{\epsilon,+}\Lambda e_{\epsilon,+}$, then $w \in J_{\epsilon,+}$.
- If i is a source and $\epsilon_i = -$, then i is an isolated vertex in the quiver of Λ_ϵ.
- Let Λ be the path algebra of $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4$ and Λ' the algebra presented by

$$\begin{array}{ccc} 1 & \longrightarrow & 3 \\ \downarrow & \ddots & \downarrow \\ 2 & \longrightarrow & 4 \end{array}.$$

  Then, $\Lambda_{(+,-,+,-)} \simeq \Lambda'_{(+,-,+,-)}$.

A reduction theorem

Theorem 2.3 (Aoki-Higashitani-Iyama-Kase-Mizuno, 2022)

For any $\epsilon \in \mathsf{s}_n$, there is an isomorphism

$$\text{2-silt}_\epsilon \Lambda \xrightarrow{\sim} \text{2-silt}_\epsilon \Lambda_\epsilon,$$

which preserves g-vectors of two-term silting complexes.

Remark: A proof first appeared in [Aoki, 2018] for radical square zero algebras.

Tilting mutation process

Fix $\epsilon_i = -$ and set $\Phi := \{\epsilon \in \mathsf{s}_n \mid \epsilon_i = -\}$. We define

$$\text{2-silt}_{\Phi}\Lambda := \bigsqcup_{\epsilon \in \Phi} \text{2-silt}_{\epsilon}\Lambda.$$

Let $\Lambda' := \operatorname{End} \mu^-_{P_i}(\Lambda)$ be the endomorphism algebra of $\mu^-_{P_i}(\Lambda)$.

Theorem 2.4 (Aoki-W, 2021)

If $\mu^-_{P_i}(\Lambda)$ is tilting, then there is a poset isomorphism

$$\text{2-silt}_{\Phi}\Lambda \xrightarrow{\sim} \text{2-silt}_{-\Phi}\Lambda'. \tag{3.1}$$

Remark: A proof first appeared in [Asashiba-Mizuno-Nakashima, 2020] for Brauer tree algebras.

Symmetry on sign decomposition

Suppose there is an algebra isomorphism $\sigma : \Lambda^{\mathrm{op}} \to \Lambda$.

Proposition 2.5

The functor S_σ gives a bijection

$$\text{2-silt}_\epsilon \Lambda \xrightarrow{1:1} \text{2-silt}_{-\sigma(\epsilon)} \Lambda,$$

where $\sigma(\epsilon) = (\epsilon_{\sigma(1)}, \epsilon_{\sigma(2)}, \ldots, \epsilon_{\sigma(n)})$.

$\Rightarrow$ We have $\Lambda_\epsilon \simeq (\Lambda_{-\sigma(\epsilon)})^{\mathrm{op}}$.

Borel-Schur algebra

- n, r: two positive integers
- K: an algebraically closed field of characteristic $p \geq 0$
- V: an n-dim vector space V with a basis $\{v_1, v_2, \ldots, v_n\}$

The tensor product $V^{\otimes r}$ admits a K-basis given by

$$\{v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r} \mid 1 \leqslant i_j \leqslant n \text{ for all } 1 \leqslant j \leqslant r\}.$$

The general linear group GL_n acts on $V^{\otimes r}$ by

$$(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r}) \cdot g = gv_{i_1} \otimes gv_{i_2} \otimes \cdots \otimes gv_{i_r}$$

for any $g \in \mathrm{GL}_n$. We then obtain a homomorphism of algebras:

$$\rho : K\mathrm{GL}_n \longrightarrow \mathrm{End}_K(V^{\otimes r}).$$

The image of ρ, i.e., $\rho(K\mathrm{GL}_n)$, is called the Schur algebra.

Let B^+ be the Borel subgroup of GL_n consisting of all upper triangular matrices.

Definition 3.1

We call the subalgebra $\rho(B^+)$ of $\rho(K\mathrm{GL}_n)$ the Borel-Schur algebra and denote it by $S^+(n, r)$.

Some nice properties:

- $S^+(n, r)$ is a basic algebra.
- $S^+(n, r)$ has a finite global dimension.
- $S^+(n, r)$ admits an explicit formula for the multiplication.

Representation type of $S^+(n, r)$

Theorem 3.2 (Erdmann-Santana-Yudin, 2018, 2021)

The Borel-Schur algebra $S^+(n, r)$ is

- representation-finite if one of the following holds:
  - $n = 2$ and $p = 0$, or $p = 2, r \leq 3$ or $p = 3, r \leq 4$ or $p \geq 5, r \leq p$;
  - $n \geq 3$ and $r = 1$.
- tame if $n = 2, p = 3, r = 5$ or $n = 3, r = 2$.

Otherwise, $S^+(n, r)$ is wild.

2-term silting finiteness

An algebra Λ is <u>2-term silting finite</u> if 2-silt Λ is a finite set.

Proposition 3.3 (Demonet-Iyama-Reading-Reiten-Thomas, 2017, Demonet-Iyama-Jasso, 2019)

If Λ is 2-term silting finite, then

(1) the quotient algebra Λ/I is 2-term silting finite, for any two-sided ideal I of Λ,

(2) the idempotent truncation $e\Lambda e$ is 2-term silting finite, for any idempotent e of Λ.

Proposition 3.4 (Mousavand, 2019)

A concealed algebra of type $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{D}}_n (n \geqslant 4)$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ or $\widetilde{\mathbb{E}}_8$ is 2-term silting infinite.

2-term silting infinite $S^+(n, r)$

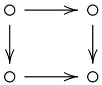
Proposition 3.5 (Erdmann-Santana-Yudin, 2021)

Let $m \leq n$ and $s \leq r$. Then,

$$S^+(m, s) = eS^+(n, r)e$$

for an idempotent e of $S^+(n, r)$.

For any $p \geq 0$, $S^+(3, 2)$ contains the path algebra of the quiver:



as an idempotent truncation. It is a concealed algebra of type $\widetilde{\mathbb{A}}_3$.

Quiver and relations of $S^+(2,r)$

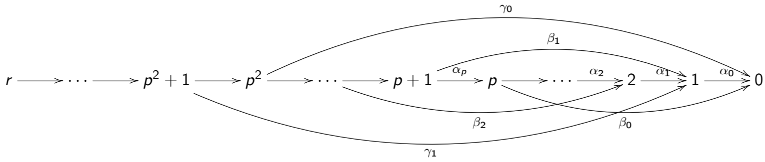
Proposition 3.6 (Erdmann-Santana-Yudin, 2018)

Let $p=0$. Then, $S^+(2,r)$ is isomorphic to the path algebra of

$$r \longrightarrow \cdots \longrightarrow 2 \longrightarrow 1 \longrightarrow 0\ .$$

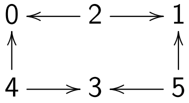
Proposition 3.7 (Liang, 2007)

Let $p>0$. Then, $S^+(2,r) \simeq K\Delta_r/\mathcal{I}$.



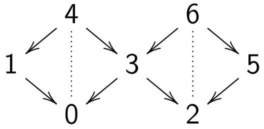
2-term silting infinite $S^+(2,r)$

- $S^+(2,5)$ over $p=2$ contains a concealed algebra of type $\widetilde{\mathbb{A}}_5$,



as a quotient algebra.

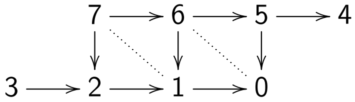
- $S^+(2,6)$ over $p=3$ contains a concealed algebra of type $\widetilde{\mathbb{D}}_6$,



as a quotient algebra.

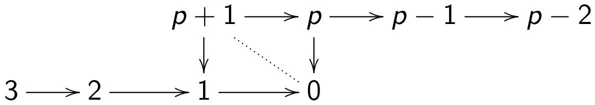
2-term silting infinite $S^+(2, r)$

- $S^+(2, 7)$ over $p = 5$ contains a concealed algebra of type $\widetilde{\mathbb{E}}_7$,



  as a quotient algebra.

- $S^+(2, p + 1)$ over $p \geq 7$ contains a concealed algebra of type $\widetilde{\mathbb{E}}_7$,



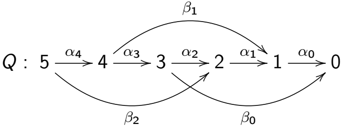
  as a quotient algebra.

2-term silting finite $S^+(2, r)$

Only three cases are remaining and they are 2-term silting finite.

- $S^+(2, 4)$ over $p = 2$.
- $S^+(2, 5)$ over $p = 3$.
- $S^+(2, 6)$ over $p = 5$.

e.g., $S^+(2, 5)$ over $p = 3$ is isomorphic to $A := KQ/I$ with



$$I : \left\langle \begin{matrix} \alpha_4\alpha_3\alpha_2, \alpha_3\alpha_2\alpha_1, \alpha_2\alpha_1\alpha_0, \\ \alpha_4\beta_1 - \beta_2\alpha_1, \alpha_3\beta_0 - \beta_1\alpha_0 \end{matrix} \right\rangle.$$

We take $\epsilon = (\epsilon_0, \epsilon_1, \ldots, \epsilon_5) \in \mathsf{s}_6$.

If $\epsilon_0 = +$ or $\epsilon_5 = -$, we have $A_\epsilon \simeq (K \oplus S^+(2,4))_\epsilon$, and then A_ϵ is 2-term silting finite.

Suppose $\epsilon_0 = -$ and $\epsilon_5 = +$. Since there is an algebra isomorphism $\sigma : A^{\mathrm{op}} \to A$ sending e_i^* to e_{5-i}, it suffices to consider

$$\begin{aligned}
&(-,+,+,-,+,+) \sim_\sigma (-,-,+,-,-,+), \quad (-,-,-,-,-,+) \sim_\sigma (-,+,+,+,+,+),\\
&(-,-,+,-,+,+), \quad (-,+,+,-,-,+), \quad (-,+,+,+,-,+) \sim_\sigma (-,+,-,-,-,+),\\
&(-,+,-,+,-,+), \quad (-,-,+,+,-,+) \sim_\sigma (-,+,-,-,+,+), \quad (-,-,-,+,+,+),\\
&(-,-,-,-,+,+) \sim_\sigma (-,-,+,+,+,+), \quad (-,-,-,+,-,+) \sim_\sigma (-,+,-,+,+,+).
\end{aligned}$$

Here, $\epsilon \sim_\sigma \epsilon'$ stands for $\epsilon' = -\sigma(\epsilon)$.

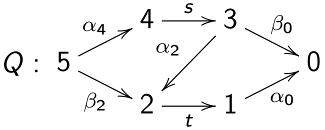
e.g.,

- $\epsilon = (-,+,+,-,+,+),(-,-,-,-,-,+),(-,-,+,-,+,+)$. Then, A_ϵ is presented as

$$\begin{array}{ccccc} 5 & \longrightarrow & 4 & \longrightarrow & 3 \\ \downarrow & & \downarrow & & \downarrow \\ 2 & \longrightarrow & 1 & \longrightarrow & 0 \end{array},$$

  which is a representation-finite simply connected algebra.

- $\epsilon = (-,+,-,+,-,+)$. Then, $A_\epsilon \simeq B_\epsilon$, where $B := KQ/I$ is given by



$I : \langle \alpha_4 s \beta_0 - \beta_2 t \alpha_0, s\alpha_2, \alpha_2 t \rangle$.

  Here, B is a representation-finite special biserial algebra.

2-term silting finiteness of $S^+(n, r)$

Theorem 3.8 (W., 2023)

The Borel-Schur algebra $S^+(n, r)$ is 2-term silting finite for

- all representation-finite cases,
- a tame case: $S^+(2, 5)$ over $p = 3$,
- two wild cases: $S^+(2, 4)$ over $p = 2$ and $S^+(2, 6)$ over $p = 5$.

Otherwise, $S^+(n, r)$ is 2-term silting infinite.

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Thank you! Any questions?

- Silting complex and silting mutation
- A partial order on $\operatorname{silt}\Lambda$
- 2-term silting complex and g-vector
- Symmetry on $\mathcal{H}(\operatorname{silt}\Lambda)$

- Structure of $2\text{-}\operatorname{silt}_\epsilon \Lambda$
- Reduction from $2\text{-}\operatorname{silt}\Lambda$ to $2\text{-}\operatorname{silt}_\epsilon \Lambda_\epsilon$
- Tilting mutation process
- Symmetry on sign decomposition
- Borel-Schur algebra
- 2-term silting finiteness of $S^+(n, r)$