Introduction to Quiver Representation Theory

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Outline

Introduction

Algebra

A finite-dimensional algebra $\Lambda = (V, \mathbb{k}, +, \cdot, \times)$:

- $(V, \mathbb{k}, +, \cdot)$ is a vector space over $\mathbb{k}$ with $\dim_{\mathbb{k}} V < \infty$.

Example (e.g., $\mathbb{k} = \mathbb{C}$)

Algebra

A finite-dimensional algebra $\Lambda = (V, \Bbbk, +, \cdot, \times)$:

- $(V, \Bbbk, +, \cdot)$ is a vector space over $\Bbbk$ with $\dim_{\Bbbk} V < \infty$.
- $\times$ is a multiplication on V (compatible with $+$ and $\cdot$) such that $a \times (b \times c) = (a \times b) \times c$, for $a, b, c \in V$.

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Example (e.g., $\mathbb{k} = \mathbb{C}$)

(1) $\mathbb{C}[x]/(x^n) = \mathrm{span}\{1, x, x^2, \cdots, x^{n-1}\}$ is an algebra.

(2) $T_3 = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \mid a_{ij} \in \mathbb{C} \right\}$ is an algebra.

Representation of algebras

A representation: Λ act on a vector space M

$$M \star \Lambda \longrightarrow M$$

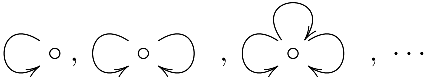
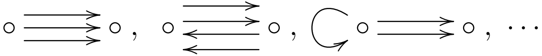
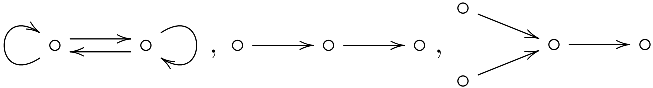
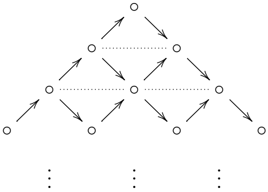
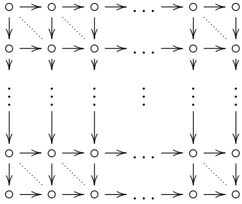
such that $(1, \lambda \in \Lambda, m, n \in M, x \in \mathbb{C})$

- $m \star 1 = m$.
- $(mx) \star \lambda = m \star (x\lambda) = (m \star \lambda)x$.
- $m \star (\lambda_1 \times \lambda_2) = (m \star \lambda_1) \star \lambda_2$.
- $(m + n) \star \lambda = m \star \lambda + n \star \lambda$.
- $m \star (\lambda_1 + \lambda_2) = m \star \lambda_1 + m \star \lambda_2$.

*A representation of Λ is also called a (right) Λ-module.

Quiver

Vertex, Arrow, Path, Cycle, Loop.

- 
- 
- 
-  

Representation of quivers

e.g., $\circ \longrightarrow \circ \longrightarrow \circ$. A representation:

$$V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3$$

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Some representations:

$$\mathbb{C} \xrightarrow{0} 0 \xrightarrow{0} 0 \qquad \mathbb{C} \xrightarrow{1} \mathbb{C} \xrightarrow{0} 0$$

$$0 \xrightarrow{0} \mathbb{C} \xrightarrow{0} 0 \qquad 0 \xrightarrow{0} \mathbb{C} \xrightarrow{1} \mathbb{C}$$

$$0 \xrightarrow{0} 0 \xrightarrow{0} \mathbb{C} \qquad \mathbb{C} \xrightarrow{1} \mathbb{C} \xrightarrow{1} \mathbb{C}$$

Representation of quivers

e.g., $\circ \longrightarrow \circ \longrightarrow \circ$. A representation:

$$V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3$$

Some representations with (some) morphisms:

$$\begin{array}{ccccc}
\mathbb{C} & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\
0 & \xrightarrow{0} & \mathbb{C} & \xrightarrow{0} & 0 \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\
0 & \xrightarrow{0} & 0 & \xrightarrow{0} & \mathbb{C}
\end{array}
\qquad\qquad
\begin{array}{ccccc}
\mathbb{C} & \xrightarrow{1} & \mathbb{C} & \xrightarrow{0} & 0 \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\
0 & \xrightarrow{0} & \mathbb{C} & \xrightarrow{1} & \mathbb{C} \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\
\mathbb{C} & \xrightarrow{1} & \mathbb{C} & \xrightarrow{1} & \mathbb{C}
\end{array}$$

e.g., $\circ \longrightarrow \circ \longrightarrow \circ$. A quiver representation:

$$V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3 .$$

- **Algebraic viewpoint**:

- **Geometric viewpoint**:

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- **Algebraic viewpoint**: to find all building blocks for quiver rep's.

- **Geometric viewpoint**:

e.g., $\circ \longrightarrow \circ \longrightarrow \circ$. A quiver representation:

$$V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3 \ .$$

- **Algebraic viewpoint**: to find all building blocks for quiver rep's. e.g., the above example has 6 building blocks:

$$\mathbb{C} \xrightarrow{0} 0 \xrightarrow{0} 0 \qquad \mathbb{C} \xrightarrow{1} \mathbb{C} \xrightarrow{0} 0$$

$$0 \xrightarrow{0} \mathbb{C} \xrightarrow{0} 0 \qquad 0 \xrightarrow{0} \mathbb{C} \xrightarrow{1} \mathbb{C}$$

$$0 \xrightarrow{0} 0 \xrightarrow{0} \mathbb{C} \qquad \mathbb{C} \xrightarrow{1} \mathbb{C} \xrightarrow{1} \mathbb{C}$$

- **Geometric viewpoint**:

e.g., $\circ \longrightarrow \circ \longrightarrow \circ$. A quiver representation:

$$V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3 \ .$$

- **Algebraic viewpoint**: to find all building blocks for quiver rep's. e.g., the above example has 6 building blocks:

$$\begin{array}{cc} \mathbb{C} \xrightarrow{0} 0 \xrightarrow{0} 0 & \mathbb{C} \xrightarrow{1} \mathbb{C} \xrightarrow{0} 0 \\ 0 \xrightarrow{0} \mathbb{C} \xrightarrow{0} 0 & 0 \xrightarrow{0} \mathbb{C} \xrightarrow{1} \mathbb{C} \\ 0 \xrightarrow{0} 0 \xrightarrow{0} \mathbb{C} & \mathbb{C} \xrightarrow{1} \mathbb{C} \xrightarrow{1} \mathbb{C} \end{array}$$

- **Geometric viewpoint**: to fix all vector spaces V_i and change matrices f, g. This gives an affine module variety.

Goal of Algebraic Representation Theory

Classify all indecomposable rep's of a given quiver Q and all morphisms between them, up to isomorphism.

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Classify all indecomposable rep's of a given quiver Q and all morphisms between them, up to isomorphism.

e.g., $\circ \longrightarrow \circ \longrightarrow \circ$ is done!

$$\mathbb{C}^2 \xrightarrow{(1,1)} \mathbb{C} \xrightarrow{1} \mathbb{C} \simeq \begin{matrix} \mathbb{C} \xrightarrow{0} 0 \xrightarrow{0} 0 \\ \oplus \\ \mathbb{C} \xrightarrow{1} \mathbb{C} \xrightarrow{1} \mathbb{C} \end{matrix}$$

Bound quiver algebras

e.g.,

$$T_3 = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \mid a_{ij} \in \mathbb{C} \right\}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

e.g.,

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$$\begin{array}{cccccc} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & \alpha & \beta & \alpha\beta \end{array}$$

We have $T_3 \simeq \mathbb{C}Q$ with $Q: \ 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$.

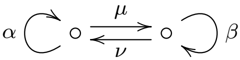
Bound quiver algebras

Any algebra Λ over $\mathbb{k}$ (e.g., $\mathbb{k} = \mathbb{C}$) is isomorphic to a **bound quiver algebra** $\mathbb{k}Q/I$. Here,

$$I = \text{span}\{\sum \lambda_i \omega_i, \cdots\},$$

$\lambda_i \in \mathbb{k}$ and ω_i is a path but not an arrow.

e.g.,



- paths: $(\alpha\mu\beta\nu)^m, (\mu\nu)^n\alpha^k, (\alpha\mu\nu)^k(\mu\beta\nu)^m$, ...

Research Area I

Algebras with nice Q in $\Bbbk Q/I$.

(1) local algebras, two-point algebras

(2) simply connected algebras

(3) Nakayama algebras, preprojective algebras

(4) special biserial algebras, string algebras, gentle algebras

Algebras with nice I in $\Bbbk Q/I$.

(1) monomial algebras

(2) incidence algebras

(3) quadratic algebras

Representation type of algebras

Theorem (Drozd 1977)

The representation type of any algebra (over $\Bbbk$) is exactly one of rep-finite, tame and wild.

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The representation type of any algebra (over $\mathbb{k}$) is exactly one of rep-finite, tame and wild.

An algebra A is said to be

- **rep-finite** if the number of indecomposable rep's is finite.
- **tame** if it is not rep-finite, but all indecomposable rep's can be organized in a one-parameter family in each dimension.

Otherwise, A is called **wild**.

Example: a tame algebra

e.g., $\circ \rightrightarrows \circ$ is tame. Indecomposable rep's:

dimension 2: $\mathbb{C} \underset{0}{\overset{1}{\rightrightarrows}} \mathbb{C}$ $\qquad$ $\mathbb{C} \underset{\lambda}{\overset{1}{\rightrightarrows}} \mathbb{C}$

dimension 3: $\mathbb{C}^2 \underset{(0,1)}{\overset{(1,0)}{\rightrightarrows}} \mathbb{C}$ $\qquad$ $\mathbb{C} \underset{(0,1)^t}{\overset{(1,0)^t}{\rightrightarrows}} \mathbb{C}^2$

dimension 4: $\mathbb{C}^2 \underset{J_2(0)}{\overset{I_2}{\rightrightarrows}} \mathbb{C}^2$ $\qquad$ $\mathbb{C}^2 \underset{J_2(\lambda)}{\overset{I_2}{\rightrightarrows}} \mathbb{C}^2$

$\vdots$

$\mathbb{C}^{n+1} \underset{[O,I_n]}{\overset{[I_n,O]}{\rightrightarrows}} \mathbb{C}^n$ $\qquad$ $\mathbb{C}^n \underset{J_n(\lambda)}{\overset{I_n}{\rightrightarrows}} \mathbb{C}^n$

Example: a wild algebra

e.g., $\circ \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} \circ$. Indecomposable rep's:

$$\text{dimension 3:} \qquad \mathbb{C}^2 \overset{(1,0)}{\underset{(0,1)}{\overset{\xrightarrow{a}}{\longrightarrow}}} \mathbb{C} \qquad a = (x, y)$$

Impossible! to give a complete classification of indecomposable rep's for a wild algebra.

Research Area II

Classify representation type of algebras. For example,

(1) local algebras, e.g., [Heller-Reiner, 1961], [Drozd, 1972], [Ringel, 1975].

(2) two-point algebras, e.g., [Bongartz-Gabriel, 1981], [Han, 2002].

(3) symmetric algebras, e.g., [Bocian-Skowronski, 2005].

(4) Hecke algebras, e.g., [Ariki, 2000], [Ariki-Mathas, 2002].

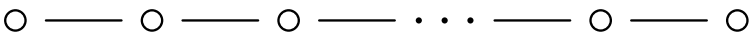
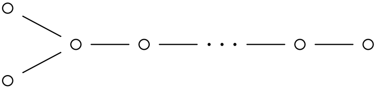
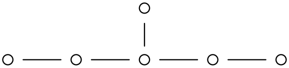
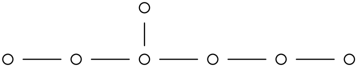
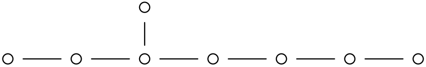
(5) q-Schur algebras, e.g., [Erdmann, 1993], [Erdmann-Nakano, 2001].

(6) block algebras of category $\mathcal{O}$; [Futorny-Nakano-Pollack, 1999].

Gabriel's Theorem

Theorem (Gabriel, 1972)

Let $\Lambda = \mathbb{k}Q$. Then, Λ is rep-finite if and only if the underlying graph of Q is one of Dynkin graphs.

- $\mathbb{A}_n$: 
- $\mathbb{D}_n$: 
- $\mathbb{E}_6$: 
- $\mathbb{E}_7$: 
- $\mathbb{E}_8$: 

e.g., set $\Lambda_n = \mathbb{k}Q/I_n$ with

$$Q: \circ \xrightarrow{\alpha} \circ \circlearrowright \beta \quad \text{and } I_n = \text{span}\{\beta^n, \alpha\beta^2\},\ n \geqslant 2.$$

The representation type of Λ_n is

- rep-finite if $n \leqslant 5$;
- tame if $n = 6$;
- wild if $n \geqslant 7$.

Auslander-Reiten Theory

e.g., $1 \longrightarrow 2 \longrightarrow 3$. We denote

$$\mathbb{C} \xrightarrow{0} 0 \xrightarrow{0} 0 \Rightarrow 1 \qquad \mathbb{C} \xrightarrow{1} \mathbb{C} \xrightarrow{0} 0 \Rightarrow \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$$

$$0 \xrightarrow{0} \mathbb{C} \xrightarrow{0} 0 \Rightarrow 2 \qquad 0 \xrightarrow{0} \mathbb{C} \xrightarrow{1} \mathbb{C} \Rightarrow \begin{smallmatrix} 2 \\ 3 \end{smallmatrix}$$

$$0 \xrightarrow{0} 0 \xrightarrow{0} \mathbb{C} \Rightarrow 3 \qquad \mathbb{C} \xrightarrow{1} \mathbb{C} \xrightarrow{1} \mathbb{C} \Rightarrow \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}$$

e.g., $1 \longrightarrow 2 \longrightarrow 3$. We denote

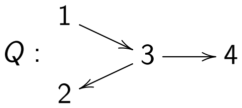
$$\mathbb{C} \xrightarrow{0} 0 \xrightarrow{0} 0 \rightsquigarrow 1 \qquad \mathbb{C} \xrightarrow{1} \mathbb{C} \xrightarrow{0} 0 \rightsquigarrow \begin{smallmatrix}1\\2\end{smallmatrix}$$

$$0 \xrightarrow{0} \mathbb{C} \xrightarrow{0} 0 \rightsquigarrow 2$$

$$0 \xrightarrow{0} 0 \xrightarrow{0} \mathbb{C} \rightsquigarrow 3$$

$$\begin{CD} 0 @>0>> \mathbb{C} @>1>> \mathbb{C} @. \rightsquigarrow \begin{smallmatrix}2\\3\end{smallmatrix} \\ @V0VV @V1VV @V1VV \\ \mathbb{C} @>1>> \mathbb{C} @>1>> \mathbb{C} @. \rightsquigarrow \begin{smallmatrix}1\\2\\3\end{smallmatrix} \end{CD}$$

Let $\Lambda = \mathbb{k}Q$ be an algebra. e.g.,



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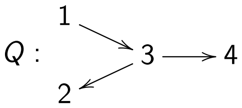
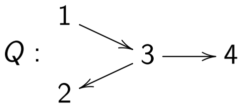


- Projective representation $P_i := e_i\Lambda$. e.g.,

$$P_1 = \begin{array}{ccc} & 1 & \\ & 3 & \\ 2 & & 4 \end{array} \qquad P_2 = 2 \qquad P_3 = \begin{array}{ccc} & 3 & \\ 2 & & 4 \end{array} \qquad P_4 = 4$$

Let $\Lambda = \mathbb{k}Q$ be an algebra. e.g.,



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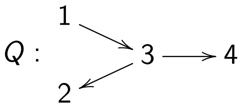
- Injective representation $I_i := \Lambda e_i$. e.g.,

$$I_1 = 1 \qquad I_2 = \begin{smallmatrix} 1 \\ 3 \\ 2 \end{smallmatrix} \qquad I_3 = \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \qquad P_4 = \begin{smallmatrix} 1 \\ 3 \\ 4 \end{smallmatrix}$$

Let $\Lambda = \mathbb{k}Q$ be an algebra. e.g.,



- Projective representation $P_i := e_i\Lambda$. e.g.,

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Set $\nu(P_i) = I_i$. (This is called Nakayama functor.)

Auslander-Reiten Translation

Let M be a representation of Λ. Take a minimal projective presentation

$$P'' \longrightarrow P' \longrightarrow M \longrightarrow 0,$$

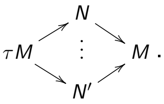
the Auslander-Reiten translation τM is defined by

$$0 \longrightarrow \tau M \longrightarrow \nu(P'') \longrightarrow \nu(P').$$

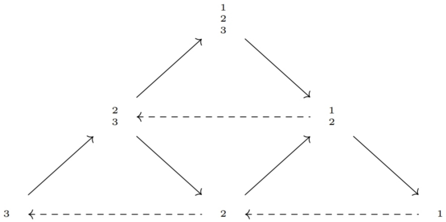
e.g., $\tau\binom{3}{2} = 4$.

Auslander-Reiten Quiver

The Auslander-Reiten quiver of an algebra Λ is defined by



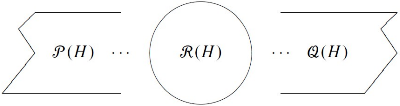
e.g., the AR-quiver of $\mathbb{k}(\ 1 \longrightarrow 2 \longrightarrow 3\ )$ is



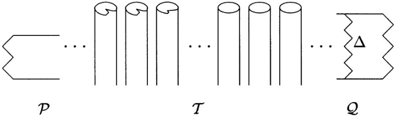
Research Area III

Find the shape of AR-quivers.

(1) If $\Lambda = \Bbbk Q$, the shape is



(2) If Λ is a tubular algebra, the shape is



Application in Quantum Groups

Cyclotomic quiver Hecke algebras

The cyclotomic quiver Hecke algebra $R^{\Lambda}(\beta)$ is defined over some Lie theoretic data $(A, P, \Pi, P^{+}, Q^{+})$.

Cyclotomic quiver Hecke algebras

The cyclotomic quiver Hecke algebra $R^{\Lambda}(\beta)$ is defined over some Lie theoretic data $(A, P, \Pi, P^{+}, Q^{+})$. In general,

$$\Lambda = a_0\Lambda_0 + a_1\Lambda_1 + \cdots + a_\ell\Lambda_\ell \in P^{+}, \; a_i \in \mathbb{Z}_{\geq 0}.$$

$$\beta = b_0\alpha_0 + b_1\alpha_1 + \cdots + b_\ell\alpha_\ell \in Q^{+}, \; b_i \in \mathbb{Z}_{\geq 0}.$$

Cyclotomic quiver Hecke algebras

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$$\beta = b_0\alpha_0 + b_1\alpha_1 + \cdots + b_\ell\alpha_\ell \in Q^+, \; b_i \in \mathbb{Z}_{\geq 0}.$$

The representation-type problem on $R^{\Lambda}(\beta)$ could be reduced to

$$\Lambda - \beta \in \{\mu - m\delta \mid \mu \in \max^+(\Lambda), m \in \mathbb{Z}_{\geq 0}\},$$

where $\delta = \alpha_0 + \alpha_1 + \ldots + \alpha_\ell$.

Known results

We know the representation type of $R^{\Lambda}(\beta)$ in the following cases.

- $R^{\Lambda_0}(\beta)$ in type $A_{2\ell}^{(2)}$, see [Ariki-Park, 2014].
- $R^{\Lambda_0}(\beta)$ in type $A_{\ell}^{(1)}$, see [Ariki-Iijima-Park, 2015].
- $R^{\Lambda_0}(\beta)$ in type $C_{\ell}^{(1)}$, see [Ariki-Park, 2015].
- $R^{\Lambda_0}(\beta)$ in type $D_{\ell+1}^{(2)}$, see [Ariki-Park, 2016].
- $R^{\Lambda_0+\Lambda_s}(\beta)$ in type $A_{\ell}^{(1)}$, see [Ariki, 2017].

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- $R^{\Lambda_0}(\beta)$ in type $A_{\ell}^{(1)}$, see [Ariki-Iijima-Park, 2015].
- $R^{\Lambda_0}(\beta)$ in type $C_{\ell}^{(1)}$, see [Ariki-Park, 2015].
- $R^{\Lambda_0}(\beta)$ in type $D_{\ell+1}^{(2)}$, see [Ariki-Park, 2016].
- $R^{\Lambda_0+\Lambda_s}(\beta)$ in type $A_{\ell}^{(1)}$, see [Ariki, 2017].

In the following, we shall explain the representation type of $R^{\Lambda}(\beta)$ in type $A_{\ell}^{(1)}$, for $\Lambda = a_{i_1}\Lambda_{i_1} + a_{i_2}\Lambda_{i_2} + \cdots + a_{i_n}\Lambda_{i_n} \in P^+$.

$\max^+(\Lambda)$

Theorem (Kim-Oh-Oh 2020)

There is a bijection $\phi_\Lambda : \max^+(\Lambda) \to P^+_{cl,k}(\Lambda)$.

Set $\Lambda = a_{i_1}\Lambda_{i_1} + a_{i_2}\Lambda_{i_2} + \cdots + a_{i_n}\Lambda_{i_n} \in P^+$. We define

$$\mathrm{le}(\Lambda) = \sum a_{i_j} \quad \text{and} \quad \mathrm{ev}(\Lambda) = i_1 + i_2 + \cdots + i_n.$$

$\max^+(\Lambda)$

Theorem (Kim-Oh-Oh 2020)

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Suppose $\mathrm{le}(\Lambda) = k$. Then,

$$P^+_{cl,k}(\Lambda) = \left\{\Lambda' \in P^+ \mid \mathrm{le}(\Lambda) = \mathrm{le}(\Lambda'), \mathrm{ev}(\Lambda) \equiv_{\ell+1} \mathrm{ev}(\Lambda') \right\}.$$

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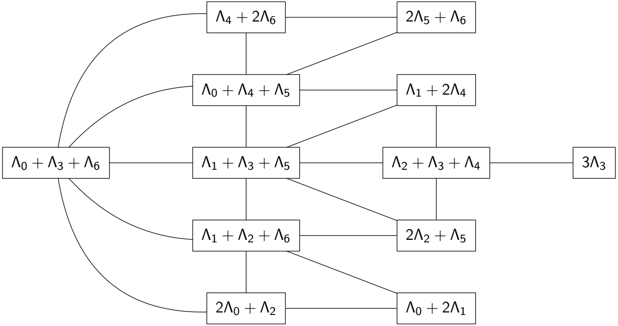
e.g., $P^+_{cl,3}(\Lambda_0 + \Lambda_3 + \Lambda_6)$ with $\ell = 6$ consists of $\Lambda_0 + \Lambda_3 + \Lambda_6$, $\Lambda_1 + \Lambda_2 + \Lambda_6$, $\Lambda_1 + \Lambda_3 + \Lambda_5$, $\Lambda_0 + \Lambda_4 + \Lambda_5$, $\Lambda_2 + \Lambda_3 + \Lambda_4$, $2\Lambda_0 + \Lambda_2$, $\Lambda_4 + 2\Lambda_6$, $2\Lambda_5 + \Lambda_6$, $\Lambda_0 + 2\Lambda_1$, $2\Lambda_2 + \Lambda_5$, $\Lambda_1 + 2\Lambda_4$, $2\Lambda_0 + \Lambda_2$, $3\Lambda_3$.

A finite connected graph

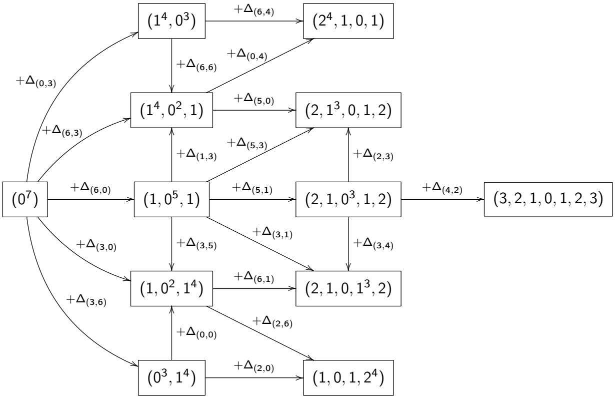
If $j \not\equiv_{\ell+1} i-1$, we draw $\boxed{\Lambda_i + \Lambda_j + \tilde{\Lambda}}$ — $\boxed{\Lambda_{i-1} + \Lambda_{j+1} + \tilde{\Lambda}}$.
e.g.,



We define

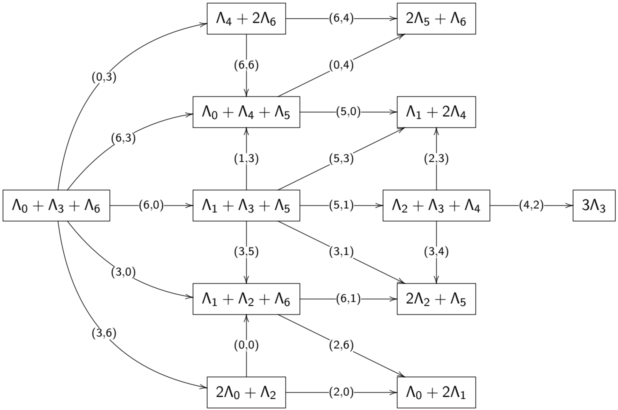
$$\Delta_{i,j} = \begin{cases} (0^i, 1^{j-i+1}, 0^{\ell-j}) & \text{if } i \leq j, \\ (1^{j+1}, 0^{i-j-1}, 1^{\ell-i+1}) & \text{if } i > j. \end{cases}$$

e.g.,



A finite quiver

e.g., $\vec{C}(\Lambda_0 + \Lambda_3 + \Lambda_6)$ with $\ell = 6$ is displayed as



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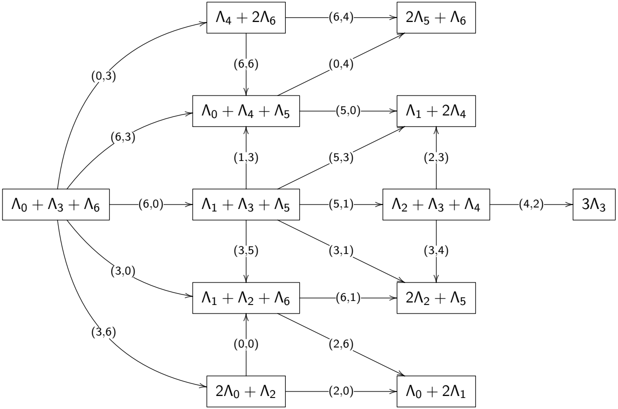
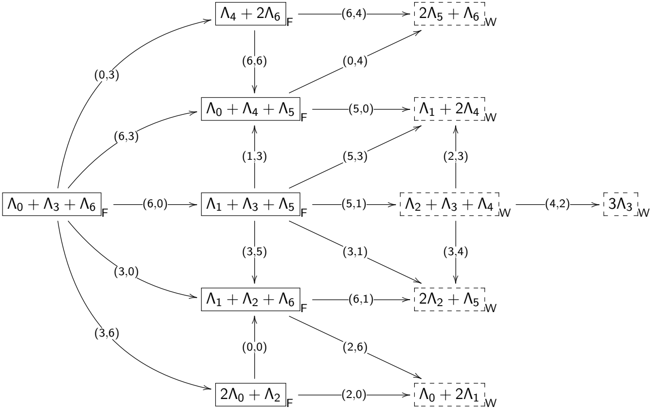


- Advantage: rep-infinite → rep-infinite   wild → wild

[Ariki-Song-W., 2023]: e.g., rep-type of $\vec{C}(\Lambda_0+\Lambda_3+\Lambda_6)$ with $\ell=6$ is displayed as



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Any questions?

Objects:
- Algebra and its representation
- Quiver and its representation
- Bound quiver algebra
- Representation type of algebra
- Gabriel's Theorem
- Projective and injective representations
- Auslander-Reiten quiver

Thank you for listening!