Representation type of cyclotomic quiver Hecke algebras[1]

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[1]Collaborations with Susumu Ariki, Berta Hudak, and Linliang Song.

Outline

Background

KLR algebras

Maximal weights

References

Background

Categorification

$$\{\text{Elements in a set}\} \xleftrightarrow{1:1} \{\text{Objects in a category}\}$$

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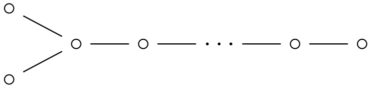
e.g., the Gabriel's Theorem gives

$$\left\{ \begin{array}{c} \text{Positive roots} \\ \text{in type } \mathbb{A}, \mathbb{D}, \mathbb{E} \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \text{Indecomposable modules} \\ \text{of the path algebra} \\ \text{in type } \mathbb{A}, \mathbb{D}, \mathbb{E} \end{array} \right\}$$

- $\mathbb{A}_n$: $\circ - \circ - \circ - \cdots - \circ - \circ$
- $\mathbb{D}_n$: 
- $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$

Cyclotomic Hecke algebra

a.k.a. Ariki-Koike algebra

- $\mathfrak{g}$: a certain Kac-Moody algebra
- Λ: a dominant integral weight for $\mathfrak{g}$
- $V(\Lambda)$: the irreducible highest weight module over $\mathfrak{g}$
- $\mathcal{H}^{\Lambda}$: the cyclotomic Hecke algebra associated with Λ

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Lie Theory	**Representation Theory**
Weight spaces of $V(\Lambda)$	Blocks of $\mathcal{H}^\Lambda$
Crystal graph of $V(\Lambda)$	Socle branching rule for $\mathcal{H}^\Lambda$
Canonical basis in $V(\Lambda)$ over $\mathbb{C}$	Indecom. projective $\mathcal{H}^\Lambda$-modules
Action of the Weyl group of $\mathfrak{g}$ on $V(\Lambda)$	Derived equivalences between blocks of $\mathcal{H}^\Lambda$

One then wants to

- draw the quantized enveloping algebra $U_q(\mathfrak{g})$ into the picture;
- give a grading on cyclotomic Hecke algebras.

This motivates the study of cyclotomic **quiver** Hecke algebras (a.k.a. cyclotomic Khovanov-Lauda-Rouquier algebras).

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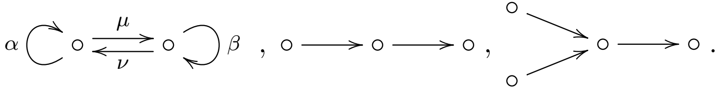
This motivates the study of cyclotomic **quiver** Hecke algebras (a.k.a. cyclotomic Khovanov-Lauda-Rouquier algebras).

$$\begin{aligned}
&\{\text{Group algebras of symmetric groups}\}\\
\subseteq\ &\{\text{Hecke algebras of type } \mathbb{A}, \mathbb{B}\}\\
\subseteq\ &\{\text{Cyclotomic Hecke algebras of type } G(k,1,n)\}\\
\subseteq\ &\{\text{Cyclotomic quiver Hecke algebras of type } A_\ell^{(1)}\}
\end{aligned}$$

Quiver Representation Theory

Quivers:



- path ω: e.g., $(\alpha\mu\beta\nu)^m$, $(\mu\nu)^n\alpha^k$, $(\alpha\mu\nu)^k(\mu\beta\nu)^m$, ...

Quiver Representation Theory

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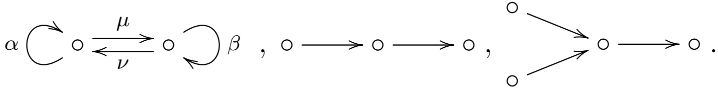

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Bound quiver algebra $A = KQ/I$:

$$I = \langle \sum \lambda_i \omega_i, \cdots \rangle$$

Representation type of algebra

Trichotomy Theorem (Drozd, 1977)

The representation type of an algebra A (over K) is exactly one of rep-finite, tame and wild.

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An algebra A is said to be

- **rep-finite** if the number of indecomposable modules is finite.
- **tame** if A is not rep-finite, but all indecomposable modules can be organized in a one-parameter family in each dimension.
- **wild** if there exists a faithful exact K-linear functor from the module category of $K\langle x, y\rangle$ to mod A.

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"The representation type of symmetric algebras is preserved under derived equivalence." *(Rickard 1991, Krause 1998)*

Preview in affine type A

Main Theorem (Ariki-Song-W., 2023)

Suppose $|\Lambda| \geq 3$. The cyclotomic quiver Hecke algebra $R^\Lambda(\beta)$ of type $A_\ell^{(1)}$ is rep-finite if $\beta \in \mathcal{F}(\Lambda)$, tame if one of the following holds:

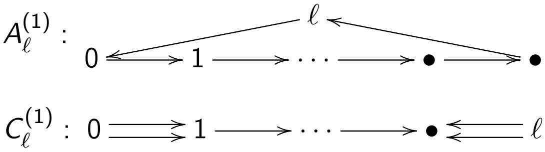
- $\beta = \delta$, $\Lambda = k\Lambda_i$, $\ell = 1$ with $t \neq \pm 2$,
- $\beta = \delta$, $\Lambda = k\Lambda_i$, $\ell \geq 2$ with $t \neq (-1)^{\ell+1}$,
- $\beta \in \mathcal{T}(\Lambda)$.

Otherwise, $R^\Lambda(\beta)$ is wild.

Cyclotomic quiver Hecke algebras

Lie theoretic data

Let $I = \{0, 1, ..., \ell\}$ be an index set. Recall that



$+B_\ell^{(1)}, D_\ell^{(1)}, A_{2\ell}^{(2)}, A_{2\ell-1}^{(2)}, D_{\ell+1}^{(2)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, F_4^{(1)}, G_2^{(1)}, E_6^{(2)}, D_4^{(3)}.$

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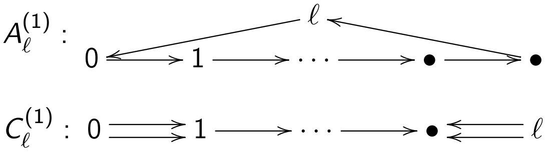

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Set $n_{ij} := \#(i \to j)$.

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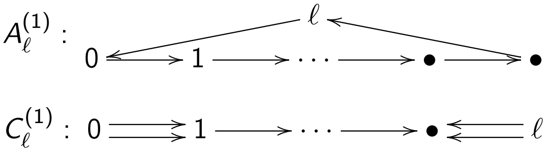

$$+B_\ell^{(1)}, D_\ell^{(1)}, A_{2\ell}^{(2)}, A_{2\ell-1}^{(2)}, D_{\ell+1}^{(2)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, F_4^{(1)}, G_2^{(1)}, E_6^{(2)}, D_4^{(3)}.$$

Set $n_{ij} := \#(i \to j)$. We define the **Cartan matrix** $\mathrm{A} = (a_{ij})_{i,j\in I}$ by

$$a_{ii} = 2, \quad a_{ij} = \begin{cases} -n_{ij} & \text{if } n_{ij} > n_{ji}, \\ -1 & \text{if } n_{ij} < n_{ji}, \\ -n_{ij} - n_{ji} & \text{otherwise,} \end{cases} \qquad (i \neq j).$$

Let $(\mathrm{A}, P, \Pi, P^\vee, \Pi^\vee)$ be the **Cartan datum** in type $X^{(1)}$, where

- $P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_\ell \oplus \mathbb{Z}\delta$ is the weight lattice;
- $\Pi = \{\alpha_i \mid 0 \leq i \leq \ell\} \subset P$ is the set of simple roots;
- $\Pi^\vee = \{h_i \mid 0 \leq i \leq \ell\}$ is the set of simple coroots.

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We have

$$\langle h_i, \alpha_j \rangle = a_{ij}, \quad \langle h_i, \Lambda_j \rangle = \delta_{ij} \quad \text{for } 0 \le i, j \le \ell.$$

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The null root is δ, e.g.,

$$\delta = \begin{cases} \alpha_0 + \alpha_1 + \cdots + \alpha_\ell & \text{if } X = A_\ell, \\ \alpha_0 + 2(\alpha_1 + \cdots + \alpha_{\ell-1}) + \alpha_\ell & \text{if } X = C_\ell. \end{cases}$$

Quiver Hecke algebra

The **quiver Hecke algebra** $R(n)$ associated with $(Q_{i,j}(u,v))_{i,j\in I}$ is the K-algebra generated by

$$\{e(\nu) \mid \nu = (\nu_1, \nu_2, \dots, \nu_n) \in I^n\}, \quad \{x_i \mid 1 \leq i \leq n\}, \quad \{\psi_j \mid 1 \leq j \leq n-1\},$$

subject to the following relations:

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subject to the following relations:

(1) $e(\nu)e(\nu') = \delta_{\nu,\nu'} e(\nu),\ \sum_{\nu\in I^n} e(\nu) = 1,\ x_i x_j = x_j x_i,\ x_i e(\nu) = e(\nu) x_i.$

(2) $\psi_i e(\nu) = e(s_i(\nu))\psi_i,\ \psi_i\psi_j = \psi_j\psi_i$ if $|i-j| > 1$.

(3) $\psi_i^2 e(\nu) = Q_{\nu_i,\nu_{i+1}}(x_i, x_{i+1})e(\nu).$

(4) $$(\psi_i x_j - x_{s_i(j)}\psi_i)e(\nu) = \begin{cases} -e(\nu) & \text{if } j = i \text{ and } \nu_i = \nu_{i+1}, \\ e(\nu) & \text{if } j = i+1 \text{ and } \nu_i = \nu_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$$

(5) $$(\psi_{i+1}\psi_i\psi_{i+1} - \psi_i\psi_{i+1}\psi_i)e(\nu) = \begin{cases} \frac{Q_{\nu_i,\nu_{i+1}}(x_i,x_{i+1}) - Q_{\nu_i,\nu_{i+1}}(x_{i+2},x_{i+1})}{x_i - x_{i+2}} e(\nu) & \text{if } \nu_i = \nu_{i+2}, \\ 0 & \text{otherwise.} \end{cases}$$

Cyclotomic quiver Hecke algebras

Set

$$\Lambda = a_0\Lambda_0 + a_1\Lambda_1 + \cdots + a_\ell\Lambda_\ell, \ a_i \in \mathbb{Z}_{\geq 0}.$$

The **cyclotomic quiver Hecke algebra** $R^\Lambda(n)$ is defined as the quotient of $R(n)$ modulo the relation

$$x_1^{\langle h_{\nu_1}, \Lambda \rangle} e(\nu) = 0.$$

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Set

$$\beta = b_0\alpha_0 + b_1\alpha_1 + \cdots + b_\ell\alpha_\ell, \ \ b_i \in \mathbb{Z}_{\geq 0},$$

with $|\beta| = b_1 + \cdots + b_\ell = n$, we define

$$R^\Lambda(\beta) := e(\beta)R^\Lambda(n)e(\beta),$$

where $e(\beta) := \sum_{\nu \in I^\beta} e(\nu)$ with $I^\beta = \left\{\nu = (\nu_1, \nu_2, \ldots, \nu_n) \in I^n \mid \sum_{i=1}^{n} \alpha_{\nu_i} = \beta\right\}$.

An example

Set $\Lambda = k\Lambda_0, \ell = 2$. Then, $I = \{0, 1, 2\}$ and $R(3)$ is generated by

$$\{e(000), \cdots, e(012), \cdots, e(212), \cdots\}, \{x_1, x_2, x_3\}, \{\psi_1, \psi_2\},$$

subject to the relations.

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Set $\beta = \alpha_0 + \alpha_1 + \alpha_2$. Then, $R^\Lambda(\beta)$ is generated by

$$\{e(012), e(021), e(102), e(120), e(201), e(210)\}, \{x_1, x_2, x_3\}, \{\psi_1, \psi_2\},$$

subject to

- $e(102) = e(120) = e(201) = e(210) = 0$, $x_1^k e(012) = x_1^k e(021) = 0$;
- $\psi_1 e(012) = \psi_1 e(021) = 0$, $\psi_2 e(012) = e(021)\psi_2$;
- $x_2 e(012) = -x_1 e(012)$, $x_2 e(021) = -t x_1 e(021)$;
- $x_3^2 e(012) = t x_1^2 e(012) + (1-t) x_1 x_3 e(012)$, etc.

Properties of $R^\Lambda(\beta)$

- $R^\Lambda(\beta)$ is a symmetric algebra, see [Shan-Varagnolo-Vasserot, 2017].

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- $R^\Lambda(\beta) \sim_{\text{derived}} R^\Lambda(\beta')$ if both $\Lambda - \beta$ and $\Lambda - \beta'$ lie in

$$\{\mu - m\delta \mid \mu \in \max^+(\Lambda), m \in \mathbb{Z}_{\geq 0}\},$$

see [Chuang-Rouquier, 2008].

$\max^+(\Lambda)$

Theorem (Kim-Oh-Oh, 2020)

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Set $\Lambda = a_{i_1}\Lambda_{i_1} + a_{i_2}\Lambda_{i_2} + \cdots + a_{i_n}\Lambda_{i_n} \in P^+, a_{i_j} \neq 0$. Then,

$$|\Lambda| := a_{i_1} + \cdots + a_{i_j} \quad \text{and} \quad \mathrm{ev}(\Lambda) := i_1 + \cdots + i_n.$$

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$$|\Lambda| := a_{i_1} + \cdots + a_{i_j} \quad \text{and} \quad \mathrm{ev}(\Lambda) := i_1 + \cdots + i_n.$$

In type $A_\ell^{(1)}$, we define

$$P_k^+(\Lambda) := \left\{ \Lambda' \in P^+ \mid |\Lambda| = |\Lambda'| = k, \mathrm{ev}(\Lambda) \equiv_{\ell+1} \mathrm{ev}(\Lambda') \right\}.$$

Recall that $\langle h_i, \Lambda_j \rangle = \delta_{ij}$. We define $y_i := \langle h_i, \Lambda - \Lambda' \rangle$ and

$$Y_{\Lambda'} := (y_0, y_1, \ldots, y_\ell) \in \mathbb{Z}^{\ell+1}.$$

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$$Y_{\Lambda'} := (y_0, y_1, \ldots, y_\ell) \in \mathbb{Z}^{\ell+1}.$$

Theorem (Ariki-Song-W., 2023)

The equation $\mathsf{A}X^t = Y^t_{\Lambda'}$ has a unique solution $X = (x_0, x_1, \ldots, x_\ell)$ satisfying

$$x_i \geq 0 \quad \text{and} \quad \min\{x_i\} = 0 \ (\min\{x_i - \delta\} < 0).$$

Set $\beta_{\Lambda'} := x_0\alpha_0 + x_1\alpha_1 + \cdots + x_\ell\alpha_\ell$. Then,

$$\begin{aligned} \phi_\Lambda^{-1} : P_k^+(\Lambda) &\to \max{}^+(\Lambda) \\ \Lambda' &\mapsto \Lambda - \beta_{\Lambda'}. \end{aligned}$$

Structure of $P_k^+(\Lambda)$

Constructions in affine type A

$$\Lambda' = \Lambda_i + \Lambda_j + \tilde{\Lambda} \in P_k^+(\Lambda) \Rightarrow \Lambda'_{i^-,j^+} := \Lambda_{i-1} + \Lambda_{j+1} + \tilde{\Lambda} \in P_k^+(\Lambda)$$

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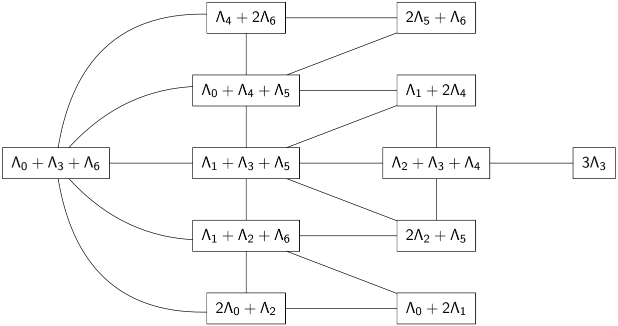
e.g., $P_3^+(\Lambda_0 + \Lambda_3 + \Lambda_6)$ in type $A_6^{(1)}$



We define

$$\Delta_{i^-,j^+} := \begin{cases} (0^i, 1^{j-i+1}, 0^{\ell-j}) & \text{if } i \leq j, \\ (1^{j+1}, 0^{i-j-1}, 1^{\ell-i+1}) & \text{if } i > j. \end{cases}$$

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We draw an arrow $\Lambda' \longrightarrow \Lambda'_{i^-,j^+}$ if

$$X_{\Lambda'} + \Delta_{i^-,j^+} = X_{\Lambda'_{i^-,j^+}}$$

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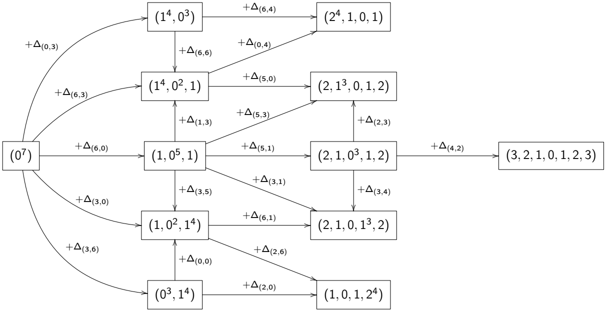

Key Lemmas

Lemma 1

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Lemma 2

Suppose $\Lambda = \bar{\Lambda} + \tilde{\Lambda}$. There is a directed path

$$\Lambda^{(1)} \longrightarrow \Lambda^{(2)} \longrightarrow \cdots \longrightarrow \Lambda^{(m)} \in \vec{C}(\bar{\Lambda})$$

if and only if there is a directed path

$$\Lambda^{(1)} + \tilde{\Lambda} \longrightarrow \Lambda^{(2)} + \tilde{\Lambda} \longrightarrow \cdots \longrightarrow \Lambda^{(m)} + \tilde{\Lambda} \in \vec{C}(\Lambda).$$

Lemma 3

Suppose that there is an arrow $\Lambda' \longrightarrow \Lambda''$ in $\vec{C}(\Lambda)$. If $R^{\Lambda}(\beta_{\Lambda'})$ is representation-infinite (resp. wild), then so is $R^{\Lambda}(\beta_{\Lambda''})$.

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Lemma 4

Write $\Lambda = \bar{\Lambda} + \tilde{\Lambda}$. If $R^{\bar{\Lambda}}(\beta)$ is representation-infinite (resp. wild), then $R^{\Lambda}(\beta)$ is representation-infinite (resp. wild).

Proof strategy in affine type A

$$\Lambda - \beta \in \{\mu - m\delta \mid \mu \in \max^+(\Lambda), m \in \mathbb{Z}_{\geq 0}\}$$

$$\Leftrightarrow \Lambda - \beta \in \{\Lambda - \beta_{\Lambda'} - m\delta \mid \Lambda' \in P_k^+(\Lambda), m \in \mathbb{Z}_{\geq 0}\}.$$

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$$\Leftrightarrow \Lambda - \beta \in \{\Lambda - \beta_{\Lambda'} - m\delta \mid \Lambda' \in P_k^+(\Lambda), m \in \mathbb{Z}_{\geq 0}\}.$$

Step 1: We show that $R^\Lambda(\beta_{\Lambda'} + m\delta)$ is wild for all $m \geq 1$ if $\beta_{\Lambda'} \neq 0$ and $R^\Lambda(m\delta)$ is wild for all $m \geq 2$, by using some **new reduction theorems**.

(If $R^\Lambda(\gamma)$ is not wild, we set $\gamma \in \mathcal{NW}(\Lambda) \cup \{\delta\}$.)

Step 2: We determine the representation type of $R^{\Lambda}(\gamma)$ for $\gamma \in \mathcal{T}(\Lambda) \cup \{\delta\}$, via case-by-case consideration.

(A systematic approach developed by Ariki and his collaborators is well applied to find the quiver presentation of $R^{\Lambda}(\gamma)$.)

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Step 3: We show that

$$\mathcal{NW}(\Lambda) \subset \mathcal{T}(\Lambda)$$

via case-by-case consideration on small k (i.e., $k = 3, 4, 5, 6$) and via induction on $k \geq 7$.

Rep-finite and tame sets in affine type A

Set $i_0 := i_h$, $i_{h+1} := i_1$ and write

$$\Lambda = m_{i_1}\Lambda_{i_1} + \cdots + m_{i_j}\Lambda_{i_j} + m_{i_{j+1}}\Lambda_{i_{j+1}} + \cdots + m_{i_h}\Lambda_{i_h}$$

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For any $1 \leq j \leq h$, we define

$$F(\Lambda)_0 := \left\{\Lambda_{i_j^-, i_j^+} \mid m_{i_j} = 2\right\}$$

$$F(\Lambda)_1 := \left\{\Lambda_{i_j^-, i_{j+1}^+} \mid m_{i_j} = 1, m_{i_{j+1}} = 1\right\}$$

$$T(\Lambda)_1 := \left\{\Lambda_{i_j^-, i_{j+1}^+} \mid m_{i_j} = 1, m_{i_{j+1}} > 1 \text{ or } m_{i_j} > 1, m_{i_{j+1}} = 1\right\}$$

$$T(\Lambda)_2 := \left\{(\Lambda_{i_j^-, i_j^+})_{(i_j-1)^-, (i_j+1)^+} \mid m_{i_j} = 2, i_{j-1} \not\equiv_{\ell+1} i_j - 1, i_{j+1} \not\equiv_{\ell+1} i_j + 1\right\} \text{ if char } K \neq 2$$

$$T(\Lambda)_3 := \left\{(\Lambda_{i_j^-, i_j^+})_{i_j^-, (i_j+1)^+ \text{ or } (i_j-1)^-, i_j^+} \mid m_{i_j} = 3, i_{j+1} \not\equiv_{\ell+1} i_j + 1 \text{ or } i_{j-1} \not\equiv_{\ell+1} i_j - 1\right\}$$
$$\text{if char } K \neq 3$$

$$T(\Lambda)_4 := \left\{(\Lambda_{i_j^-, i_j^+})_{i_j^-, i_j^+} \mid m_{i_j} = 4\right\} \text{ if char } K \neq 2$$

$$T(\Lambda)_5 := \left\{(\Lambda_{i_j^-, i_j^+})_{i_p^-, i_p^+} \mid m_{i_j} = m_{i_p} = 2, i_p \not\equiv_{\ell+1} i_j \pm 1, j \neq p\right\}$$

Set

$$\mathcal{F}(\Lambda) := \{\beta_{\Lambda'} \mid \Lambda' \in \{\Lambda\} \cup F(\Lambda)_0 \cup F(\Lambda)_1\},$$

$$\mathcal{T}(\Lambda) := \{\beta_{\Lambda'} \mid \Lambda' \in \cup_{1\le j\le 5} T(\Lambda)_j\}.$$

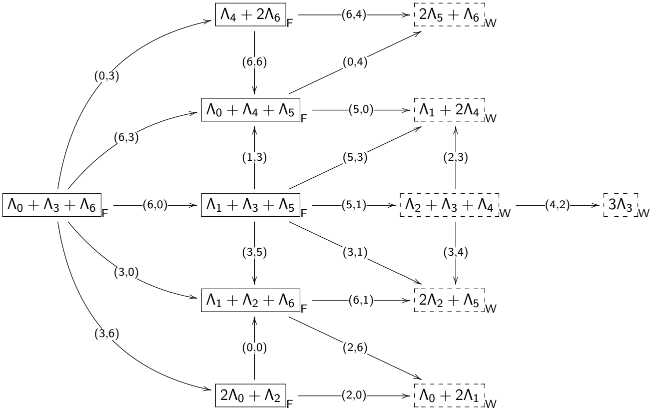
Theorem (Ariki-Song-W., 2023)

Suppose $|\Lambda| \geq 3$. Then, $R^{\Lambda}(\beta)$ is representation-finite if $\beta \in \mathcal{F}(\Lambda)$, tame if one of the following holds:

- $\beta = \delta$, $\Lambda = k\Lambda_i$, $\ell = 1$ with $t \neq \pm 2$,
- $\beta = \delta$, $\Lambda = k\Lambda_i$, $\ell \geq 2$ with $t \neq (-1)^{\ell+1}$,
- $\beta \in \mathcal{T}(\Lambda)$.

Otherwise, it is wild.

e.g., rep-type of $\vec{C}(\Lambda_0+\Lambda_3+\Lambda_6)$ in type $A_6^{(1)}$ is displayed as



Rule to draw arrows

Let Δ^+_{fin} be the set of positive roots of the root system of type X.

- If $X = A_\ell$, $\Delta^+_{\mathrm{fin}} = \{\epsilon_i - \epsilon_j \mid 1 \le i < j \le \ell + 1\}$.
- If $X = B_\ell$, $\Delta^+_{\mathrm{fin}} = \{\epsilon_i \mid 1 \le i \le \ell\} \sqcup \{\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le \ell\}$.
- If $X = C_\ell$, $\Delta^+_{\mathrm{fin}} = \{2\epsilon_i \mid 1 \le i \le \ell\} \sqcup \{\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le \ell\}$.
- If $X = D_\ell$, $\Delta^+_{\mathrm{fin}} = \{\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le \ell\}$.

Rule to draw arrows

Let Δ_{fin}^+ be the set of positive roots of the root system of type X.

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- If $X = C_\ell$, $\Delta_{\mathrm{fin}}^+ = \{2\epsilon_i \mid 1 \le i \le \ell\} \sqcup \{\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le \ell\}$.
- If $X = D_\ell$, $\Delta_{\mathrm{fin}}^+ = \{\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le \ell\}$.

Then, the set $\Delta_{\mathrm{fin}}^+ \sqcup (\delta - \Delta_{\mathrm{fin}}^+)$ gives all arrows $\Lambda' \longrightarrow \Lambda''$.

Arrows in affine type A

Recall that $\delta = \alpha_0 + \alpha_1 + \cdots + \alpha_\ell = (1, 1, \dots, 1)$. Then,

$$\Delta^+_{\mathrm{fin}} \sqcup (\delta - \Delta^+_{\mathrm{fin}}) = \{\epsilon_i - \epsilon_j, \delta - (\epsilon_i - \epsilon_j) \mid 1 \le i < j \le \ell + 1\}.$$

We have $\Delta_{i^-, j^+} =$

$$\begin{cases} (0^i, 1^{j-i+1}, 0^{\ell-j}) = \epsilon_i - \epsilon_{j+1} & \text{if } 0 < i \le j \le \ell, \\ (1^{j+1}, 0^{\ell-j}) = \delta - (\epsilon_{j+1} - \epsilon_{\ell+1}) & \text{if } 0 = i \le j \le \ell - 1, \\ (1^{j+1}, 0^{i-j-1}, 1^{\ell-i+1}) = \delta - (\epsilon_{j+1} - \epsilon_i) & \text{if } 0 \le j < i \le \ell. \end{cases}$$

Arrows in affine type C

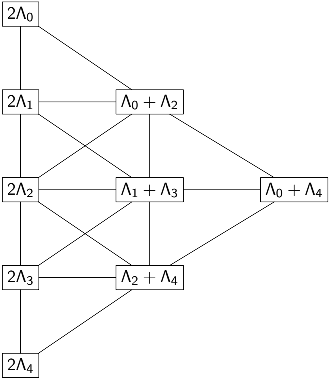
Recall that $\delta = \alpha_0 + 2\alpha_1 + \cdots + 2\alpha_{\ell-1} + \alpha_\ell = (1, 2, \ldots, 2, 1)$.

- $\Delta_{i^+} = (1, 2^i, 1, 0^{\ell-i-1}) = \delta - (\epsilon_{i+1} + \epsilon_{i+2})$.
$$\Rightarrow \{\delta - (\epsilon_i + \epsilon_{i+1}) \mid 1 \le i \le \ell - 1\}.$$
- $\Delta_{i^-} = (0^{i-1}, 1, 2^{\ell-i}, 1) = \epsilon_{i-1} + \epsilon_i$.
$$\Rightarrow \{\epsilon_i + \epsilon_{i+1} \mid 1 \le i \le \ell - 1\}.$$
- $\Delta_{i^+, j^+} = (1, 2^i, 1^{j-i}, 0^{\ell-j})$ with $i + 1 \neq j$.
$$\Rightarrow \{\delta - (\epsilon_i + \epsilon_j) \mid 1 \le i \le j \le \ell - 1, i + 1 \neq j\}.$$
- $\Delta_{i^-, j^-} = (0^i, 1^{j-i}, 2^{\ell-j}, 1)$ with $i + 1 \neq j$.
$$\Rightarrow \{\epsilon_i + \epsilon_j \mid 1 \le i \le j \le \ell - 1, i + 1 \neq j\}.$$
- Δ_{i^-, j^+} with $i \neq 0, j \neq \ell, i - 1 \neq j$.
$$\Rightarrow \{\epsilon_i - \epsilon_j, \delta - (\epsilon_i - \epsilon_j) \mid 1 \le i < j \le \ell - 1\}.$$

Constructions in affine type C

$$\Lambda' \in P_k^+(\Lambda) \Rightarrow \Lambda'_{i\pm} := \Lambda_{i\pm 2} + \tilde{\Lambda} \in P_k^+(\Lambda)$$
$$\Rightarrow \Lambda'_{i\pm, j\pm} := \Lambda_{i\pm 1} + \Lambda_{j\pm 1} + \tilde{\Lambda} \in P_k^+(\Lambda)$$

e.g., $P_2^+(2\Lambda_2)$ in type $C_4^{(1)}$



We define

- $\Delta_{i^+} := (1, 2^i, 1, 0^{\ell-i-1}), \quad \Delta_{i^-} := (0^{i-1}, 1, 2^{\ell-i}, 1).$
- $\Delta_{i^+,j^+} := (1, 2^i, 1^{j-i}, 0^{\ell-j}), \quad \Delta_{i^-,j^-} := (0^i, 1^{j-i}, 2^{\ell-j}, 1).$
- $\Delta_{i^-,j^+} := \begin{cases} (0^i, 1^{j-i+1}, 0^{\ell-j}) & \text{if } i \leq j, \\ (1, 2^j, 1^{i-j-1}, 2^{\ell-i}, 1) & \text{if } i \geq j+2. \end{cases}$

Set Δ and Λ'' for $\Lambda'_{i^\pm}$, $\Lambda'_{i^\pm,j^\pm}$, Λ'_{i^-,j^+}, respectively.

We define

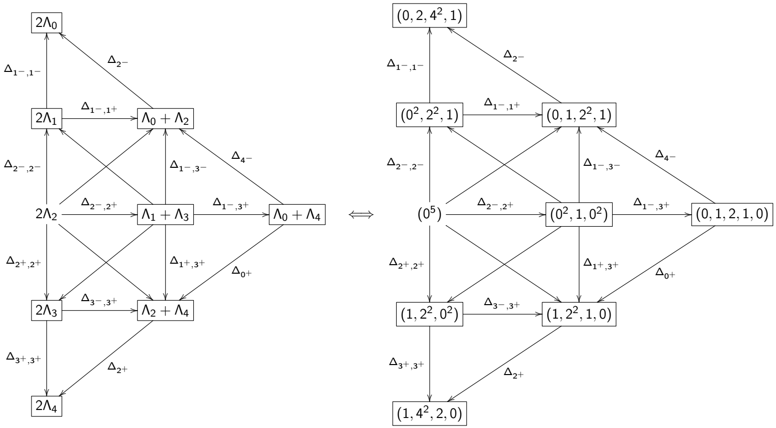
- $\Delta_{i^+} := (1, 2^i, 1, 0^{\ell-i-1}), \quad \Delta_{i^-} := (0^{i-1}, 1, 2^{\ell-i}, 1)$.
- $\Delta_{i^+,j^+} := (1, 2^i, 1^{j-i}, 0^{\ell-j}), \quad \Delta_{i^-,j^-} := (0^i, 1^{j-i}, 2^{\ell-j}, 1)$.
- $\Delta_{i^-,j^+} := \begin{cases} (0^i, 1^{j-i+1}, 0^{\ell-j}) & \text{if } i \le j, \\ (1, 2^j, 1^{i-j-1}, 2^{\ell-i}, 1) & \text{if } i \ge j+2. \end{cases}$

Set Δ and Λ'' for $\Lambda'_{i^\pm}$, $\Lambda'_{i^\pm,j^\pm}$, Λ'_{i^-,j^+}, respectively.

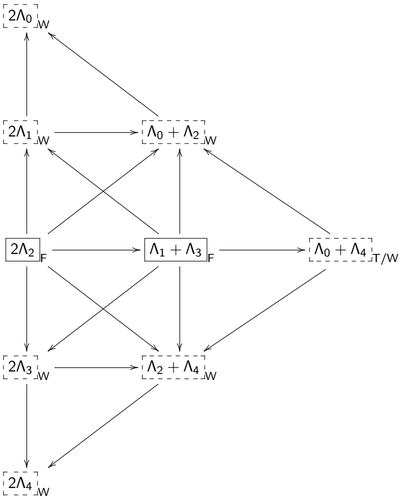
We draw an arrow $\Lambda' \longrightarrow \Lambda''$ if

$$X_{\Lambda'} + \Delta = X_{\Lambda''}.$$

e.g., the quiver for $P_2^+(2\Lambda_2)$ in type $C_4^{(1)}$ is displayed as



e.g., rep-type of $\vec{C}(2\Lambda_2)$ in type $C_4^{(1)}$ is displayed as



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Thank you! Any questions?

Background
- Categorification;
- Cyclotomic Hecke algebras;
- Bound quiver algebras;
- Representation type: rep-finite, tame, wild.

Objects
- Lie theoretic data;
- Cyclotomic KLR algebras;
- $\max^+(\Lambda)$ and $P_k^+(\Lambda)$;
- Rule to draw arrows;
- Rep-finite and tame sets in affine type A.