Representation type of cyclotomic quiver Hecke algebras[1]

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[1]Collaborations with Susumu Ariki, Berta Hudak, and Linliang Song.

Outline

Background

Categorification

$$\{\text{Elements in a set}\} \xleftrightarrow{1:1} \{\text{Objects in a category}\}$$

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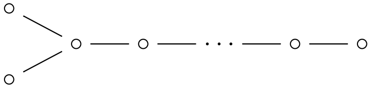
e.g., the Gabriel's Theorem gives

$$\left\{\begin{array}{c}\text{Positive roots}\\ \text{in type } \mathbb{A}, \mathbb{D}, \mathbb{E}\end{array}\right\} \xleftrightarrow{1:1} \left\{\begin{array}{c}\text{Indecomposable modules}\\ \text{of the path algebra}\\ \text{in type } \mathbb{A}, \mathbb{D}, \mathbb{E}\end{array}\right\}$$

- $\mathbb{A}_n$: $\circ - \circ - \circ - \cdots - \circ - \circ$
- $\mathbb{D}_n$: 
- $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$

Cyclotomic Hecke algebra

a.k.a. Ariki-Koike algebra

- $\mathfrak{g}$: a certain Kac-Moody algebra
- Λ: a dominant integral weight for $\mathfrak{g}$
- $V(\Lambda)$: the irreducible highest weight module over $\mathfrak{g}$
- $\mathcal{H}^\Lambda$: the cyclotomic Hecke algebra associated with Λ

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Lie Theory	Representation Theory
Weight spaces of $V(\Lambda)$	Blocks of $\mathcal{H}^\Lambda$
Crystal graph of $V(\Lambda)$	Socle branching rule for $\mathcal{H}^\Lambda$
Canonical basis in $V(\Lambda)$ over $\mathbb{C}$	Indecom. projective $\mathcal{H}^\Lambda$-modules
Action of the Weyl group of $\mathfrak{g}$ on $V(\Lambda)$	Derived equivalences between blocks of $\mathcal{H}^\Lambda$

One then wants to

- draw the quantized enveloping algebra $U_q(\mathfrak{g})$ into the picture;
- give a grading on cyclotomic Hecke algebras.

This motivates the study of cyclotomic **quiver** Hecke algebras (a.k.a. cyclotomic Khovanov-Lauda-Rouquier algebras).

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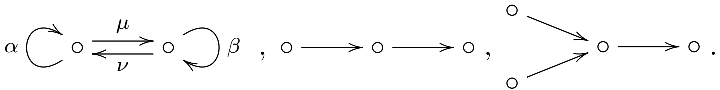
This motivates the study of cyclotomic **quiver** Hecke algebras (a.k.a. cyclotomic Khovanov-Lauda-Rouquier algebras).

$$
\begin{aligned}
&\{\text{Group algebras of symmetric groups}\}\\
&\subseteq \{\text{Hecke algebras of type } \mathbb{A}\}\\
&\subseteq \{\text{Cyclotomic Hecke algebras of type } G(k,1,n)\}\\
&\subseteq \{\text{Cyclotomic quiver Hecke algebras of type } A_\ell^{(1)}\}
\end{aligned}
$$

Quiver Representation Theory

Quivers:



- path ω: e.g., $(\alpha\mu\beta\nu)^m$, $(\mu\nu)^n\alpha^k$, $(\alpha\mu\nu)^k(\mu\beta\nu)^m$, ...

Quiver Representation Theory

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$$
\begin{tikzcd}
\circ \arrow[loop left, "\alpha"] \arrow[r, shift left, "\mu"] & \circ \arrow[l, shift left, "\nu"] \arrow[loop right, "\beta"]
\end{tikzcd}
,\quad
\begin{tikzcd}
\circ \arrow[r] & \circ \arrow[r] & \circ
\end{tikzcd}
,\quad
\begin{tikzcd}
\circ \arrow[dr] & & \\
& \circ \arrow[r] & \circ \\
\circ \arrow[ur] & &
\end{tikzcd}
.
$$

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Bound quiver algebra $A = KQ/I$:

$$I = \langle \sum \lambda_i \omega_i, \cdots \rangle$$

Representation type of algebra

Trichotomy Theorem (Drozd, 1977)

The representation type of an algebra A (over K) is exactly one of rep-finite, tame and wild.

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An algebra A is said to be

- **rep-finite** if the number of indecomposable modules is finite.
- **tame** if A is not rep-finite, but all indecomposable modules can be organized in a one-parameter family in each dimension.
- **wild** if there exists a faithful exact K-linear functor from the module category of $K\langle x, y\rangle$ to $\operatorname{mod} A$.

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"The representation type of symmetric algebras is preserved under derived equivalence." (Rickard 1991, Krause 1998)

Cyclotomic quiver Hecke algebras

Lie theoretic data

Let $I = \{0, 1, ..., \ell\}$ be an index set. Recall that

$$A_\ell^{(1)} : \begin{tikzcd} & & \ell \arrow[dll] & & \\ 0 \arrow[r] & 1 \arrow[r] & \cdots \arrow[r] & \circ \arrow[r] & \circ \arrow[ull] \end{tikzcd}$$

$$C_\ell^{(1)} : \begin{tikzcd} 0 \arrow[r, Rightarrow] & 1 \arrow[r] & \cdots \arrow[r] & \circ & \ell \arrow[l, Rightarrow] \end{tikzcd}$$

$$+B_\ell^{(1)}, D_\ell^{(1)}, A_{2\ell}^{(2)}, A_{2\ell-1}^{(2)}, D_{\ell+1}^{(2)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, F_4^{(1)}, G_2^{(1)}, E_6^{(2)}, D_4^{(3)}.$$

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Set $n_{ij} := \#(i \to j)$. We define the **Cartan matrix** $\mathrm{A} = (a_{ij})_{i,j \in I}$ by

$$a_{ii} = 2, \quad a_{ij} = \begin{cases} -n_{ij} & \text{if } n_{ij} > n_{ji}, \\ -1 & \text{if } n_{ij} < n_{ji}, \\ -n_{ij} - n_{ji} & \text{otherwise,} \end{cases} \qquad (i \neq j).$$

Cyclotomic quiver Hecke algebra

Set

$$\Lambda = a_0\Lambda_0 + a_1\Lambda_1 + \cdots + a_\ell\Lambda_\ell, \;\; a_i \in \mathbb{Z}_{\geq 0},$$

$$\beta = b_0\alpha_0 + b_1\alpha_1 + \cdots + b_\ell\alpha_\ell, \;\; b_i \in \mathbb{Z}_{\geq 0}.$$

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The **cyclotomic quiver Hecke algebra** $R^\Lambda(\beta)$ is the K-algebra generated by

$$\{e(\nu) \mid \nu = (\nu_1, \nu_2, \ldots, \nu_n) \in I^n\}, \quad \{x_i \mid 1 \leq i \leq n\}, \quad \{\psi_j \mid 1 \leq j \leq n-1\},$$

subject to some relations.

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subject to some relations.

- $R^\Lambda(\beta)$ is a symmetric algebra, see [Shan-Varagnolo-Vasserot, 2017].
- $R^\Lambda(\beta) \sim_{\text{derived}} R^\Lambda(\beta')$ if both $\Lambda - \beta$ and $\Lambda - \beta'$ lie in

$$\{\mu - k\delta \mid \mu \in \max^+(\Lambda), k \in \mathbb{Z}_{\geq 0}\}.$$

$\max^+(\Lambda)$

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Set $\Lambda = a_{i_1}\Lambda_{i_1} + a_{i_2}\Lambda_{i_2} + \cdots + a_{i_n}\Lambda_{i_n}, a_{i_j} \neq 0$. Then,

$$|\Lambda| := a_{i_1} + \cdots + a_{i_j} \quad \text{and} \quad \mathrm{ev}(\Lambda) := i_1 + \cdots + i_n.$$

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In type $A_\ell^{(1)}$, we have

$$P_k^+(\Lambda) := \left\{\Lambda' \in P^+ \mid |\Lambda| = |\Lambda'| = k, \mathrm{ev}(\Lambda) \equiv_{\ell+1} \mathrm{ev}(\Lambda')\ \right\}.$$

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In type $C_\ell^{(1)}$, we have

$$P_k^+(\Lambda) := \left\{\Lambda' \in P^+ \mid |\Lambda| = |\Lambda'| = k, \mathrm{ev}(\Lambda) \equiv_2 \mathrm{ev}(\Lambda') \right\}.$$

Recall that $\langle h_i, \Lambda_j \rangle = \delta_{ij}$. We define $y_i := \langle h_i, \Lambda - \Lambda' \rangle$ and

$$Y_{\Lambda'} := (y_0, y_1, \ldots, y_\ell) \in \mathbb{Z}^{\ell+1}.$$

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Theorem (Ariki-Song-W., 2023)

The equation $AX^t = Y^t_{\Lambda'}$ has a unique solution $X = (x_0, x_1, \ldots, x_\ell)$ satisfying

$$x_i \geq 0 \quad \text{and} \quad \min\{x_i - \delta\} < 0.$$

Set $\beta_{\Lambda'} := x_0\alpha_0 + x_1\alpha_1 + \cdots + x_\ell\alpha_\ell$. Then,

$$\begin{aligned} \phi_\Lambda^{-1} : P_k^+(\Lambda) &\to \max{}^+(\Lambda) \\ \Lambda' &\mapsto \Lambda - \beta_{\Lambda'}. \end{aligned}$$

Structure of $P_k^+(\Lambda)$

Constructions in affine type A

$$\Lambda' = \Lambda_i + \Lambda_j + \tilde{\Lambda} \in P_k^+(\Lambda) \Rightarrow \Lambda'_{i^-,j^+} := \Lambda_{i-1} + \Lambda_{j+1} + \tilde{\Lambda} \in P_k^+(\Lambda)$$

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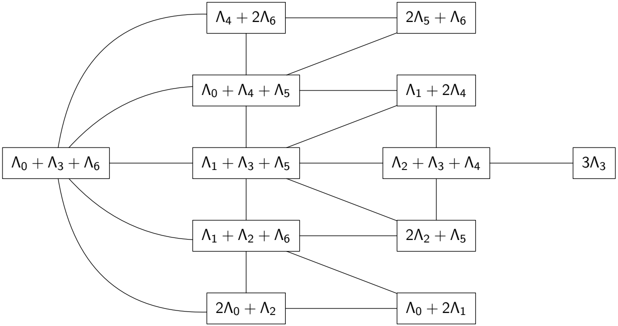
e.g., $P_3^+(\Lambda_0 + \Lambda_3 + \Lambda_6)$ in type $A_6^{(1)}$



We define

$$\Delta_{i^-,j^+} := \begin{cases} (0^i, 1^{j-i+1}, 0^{\ell-j}) & \text{if } i \leq j, \\ (1^{j+1}, 0^{i-j-1}, 1^{\ell-i+1}) & \text{if } i > j. \end{cases}$$

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We draw an arrow $\Lambda' \longrightarrow \Lambda'_{i^-,j^+}$ if

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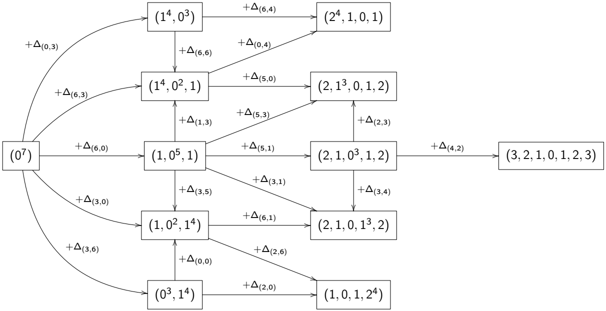
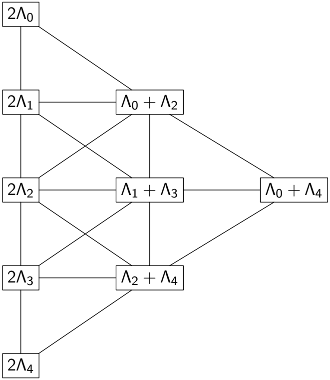


Constructions in affine type C

$$\Lambda' \in P_k^+(\Lambda) \Rightarrow \Lambda'_{i\pm} := \Lambda_{i\pm 2} + \tilde{\Lambda} \in P_k^+(\Lambda)$$
$$\Rightarrow \Lambda'_{i\pm, j\pm} := \Lambda_{i\pm 1} + \Lambda_{j\pm 1} + \tilde{\Lambda} \in P_k^+(\Lambda)$$

e.g., $P_2^+(2\Lambda_2)$ in type $C_4^{(1)}$



We define

- $\Delta_{i^+} := (1, 2^i, 1, 0^{\ell-i-1}), \quad \Delta_{i^-} := (0^{i-1}, 1, 2^{\ell-i}, 1)$.
- $\Delta_{i^+,j^+} := (1, 2^i, 1^{j-i}, 0^{\ell-j}), \quad \Delta_{i^-,j^-} := (0^i, 1^{j-i}, 2^{\ell-j}, 1)$.
- $\Delta_{i^-,j^+} := \begin{cases} (0^i, 1^{j-i+1}, 0^{\ell-j}) & \text{if } i \leq j, \\ (1, 2^j, 1^{i-j-1}, 2^{\ell-i}, 1) & \text{if } i \geq j+2. \end{cases}$

Set Δ and Λ'' for $\Lambda'_{i^\pm}$, $\Lambda'_{i^\pm,j^\pm}$, Λ'_{i^-,j^+}, respectively.

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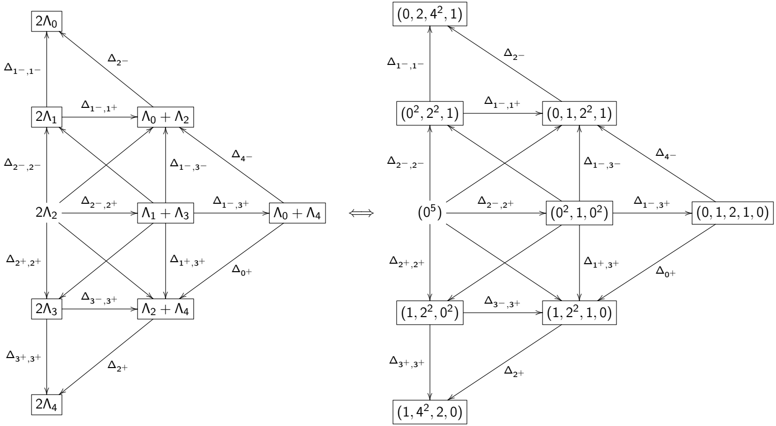
We draw an arrow $\Lambda' \longrightarrow \Lambda''$ if

$$X_{\Lambda'} + \Delta = X_{\Lambda''}.$$

e.g., the quiver for $P_2^+(2\Lambda_2)$ in type $C_4^{(1)}$ is displayed as



Rule to draw arrows

Let Δ^+_{fin} be the set of positive roots of the root system of type X.

- If $X = A_\ell$, $\Delta^+_{\mathrm{fin}} = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq \ell + 1\}$.
- If $X = B_\ell$, $\Delta^+_{\mathrm{fin}} = \{\epsilon_i \mid 1 \leq i \leq \ell\} \sqcup \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq \ell\}$.
- If $X = C_\ell$, $\Delta^+_{\mathrm{fin}} = \{2\epsilon_i \mid 1 \leq i \leq \ell\} \sqcup \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq \ell\}$.
- If $X = D_\ell$, $\Delta^+_{\mathrm{fin}} = \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq \ell\}$.

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- If $X = C_\ell$, $\Delta_{\mathrm{fin}}^+ = \{2\epsilon_i \mid 1 \leq i \leq \ell\} \sqcup \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq \ell\}$.
- If $X = D_\ell$, $\Delta_{\mathrm{fin}}^+ = \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq \ell\}$.

Then, the set $\Delta_{\mathrm{fin}}^+ \sqcup (\delta - \Delta_{\mathrm{fin}}^+)$ gives all arrows $\Lambda' \longrightarrow \Lambda''$.

Arrows in affine type A

Recall that $\delta = \alpha_0 + \alpha_1 + \cdots + \alpha_\ell = (1, 1, \ldots, 1)$. Then,

$$\Delta^+_{\mathrm{fin}} \sqcup (\delta - \Delta^+_{\mathrm{fin}}) = \{\epsilon_i - \epsilon_j, \delta - (\epsilon_i - \epsilon_j) \mid 1 \leq i < j \leq \ell + 1\}.$$

We have $\Delta_{i^-, j^+} =$

$$\begin{cases} (0^i, 1^{j-i+1}, 0^{\ell-j}) = \epsilon_i - \epsilon_{j+1} & \text{if } 0 < i \leq j \leq \ell, \\ (1^{j+1}, 0^{\ell-j}) = \delta - (\epsilon_{j+1} - \epsilon_{\ell+1}) & \text{if } 0 = i \leq j \leq \ell - 1, \\ (1^{j+1}, 0^{i-j-1}, 1^{\ell-i+1}) = \delta - (\epsilon_{j+1} - \epsilon_i) & \text{if } 0 \leq j < i \leq \ell. \end{cases}$$

Arrows in affine type C

Recall that $\delta = \alpha_0 + 2\alpha_1 + \cdots + 2\alpha_{\ell-1} + \alpha_\ell = (1, 2, \ldots, 2, 1)$.

- $\Delta_{i^+} = (1, 2^i, 1, 0^{\ell-i-1}) = \delta - (\epsilon_{i+1} + \epsilon_{i+2})$.

$$\Rightarrow \{\delta - (\epsilon_i + \epsilon_{i+1}) \mid 1 \leq i \leq \ell - 1\}.$$

- $\Delta_{i^-} = (0^{i-1}, 1, 2^{\ell-i}, 1) = \epsilon_{i-1} + \epsilon_i$.

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- $\Delta_{i^+,j^+} = (1, 2^i, 1^{j-i}, 0^{\ell-j})$ with $i + 1 \neq j$.

$$\Rightarrow \{\delta - (\epsilon_i + \epsilon_j) \mid 1 \leq i \leq j \leq \ell - 1, i + 1 \neq j\}.$$

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$$\Rightarrow \{\epsilon_i + \epsilon_j \mid 1 \leq i \leq j \leq \ell - 1, i + 1 \neq j\}.$$

- Δ_{i^-,j^+} with $i \neq 0, j \neq \ell, i - 1 \neq j$.

$$\Rightarrow \{\epsilon_i - \epsilon_j, \delta - (\epsilon_i - \epsilon_j) \mid 1 \leq i < j \leq \ell - 1\}.$$

Key Lemmas

Lemma 1

The quiver $\vec{C}(\Lambda)$ of $P_k^+(\Lambda)$ is a finite connected quiver.

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Lemma 2

Suppose $\Lambda = \bar{\Lambda} + \tilde{\Lambda}$. There is a directed path

$$\Lambda^{(1)} \longrightarrow \Lambda^{(2)} \longrightarrow \cdots \longrightarrow \Lambda^{(m)} \in \vec{C}(\bar{\Lambda})$$

if and only if there is a directed path

$$\Lambda^{(1)} + \tilde{\Lambda} \longrightarrow \Lambda^{(2)} + \tilde{\Lambda} \longrightarrow \cdots \longrightarrow \Lambda^{(m)} + \tilde{\Lambda} \in \vec{C}(\Lambda).$$

Lemma 3

Suppose that there is an arrow $\Lambda' \longrightarrow \Lambda''$ in $\vec{C}(\Lambda)$. If $R^{\Lambda}(\beta_{\Lambda'})$ is representation-infinite (resp. wild), then so is $R^{\Lambda}(\beta_{\Lambda''})$.

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Lemma 4

Write $\Lambda = \bar{\Lambda} + \tilde{\Lambda}$. If $R^{\bar{\Lambda}}(\beta)$ is representation-infinite (resp. wild), then $R^{\Lambda}(\beta)$ is representation-infinite (resp. wild).

Rep-finite and tame sets in affine type A

Set $i_0 := i_h$, $i_{h+1} := i_1$ and write

$$\Lambda = m_{i_1}\Lambda_{i_1} + \cdots + m_{i_j}\Lambda_{i_j} + m_{i_{j+1}}\Lambda_{i_{j+1}} + \cdots + m_{i_h}\Lambda_{i_h}$$

Rep-finite and tame sets in affine type A

Set $i_0 := i_h$, $i_{h+1} := i_1$ and write

$$\Lambda = m_{i_1}\Lambda_{i_1} + \cdots + m_{i_j}\Lambda_{i_j} + m_{i_{j+1}}\Lambda_{i_{j+1}} + \cdots + m_{i_h}\Lambda_{i_h}$$

For any $1 \le j \le h$, we define

$$F(\Lambda)_0 := \left\{\Lambda_{i_j^-, i_j^+} \mid m_{i_j} = 2\right\}$$
$$F(\Lambda)_1 := \left\{\Lambda_{i_j^-, i_{j+1}^+} \mid m_{i_j} = 1, m_{i_{j+1}} = 1\right\}$$
$$T(\Lambda)_1 := \left\{\Lambda_{i_j^-, i_{j+1}^+} \mid m_{i_j} = 1, m_{i_{j+1}} > 1 \text{ or } m_{i_j} > 1, m_{i_{j+1}} = 1\right\}$$
$$T(\Lambda)_2 := \left\{(\Lambda_{i_j^-, i_j^+})_{(i_j-1)^-, (i_j+1)^+} \mid m_{i_j} = 2, i_{j-1} \not\equiv_{\ell+1} i_j - 1, i_{j+1} \not\equiv_{\ell+1} i_j + 1\right\} \text{ if char } K \neq 2$$
$$T(\Lambda)_3 := \left\{(\Lambda_{i_j^-, i_j^+})_{i_j^-, (i_j+1)^+ \text{ or } (i_j-1)^-, i_j^+} \mid m_{i_j} = 3, i_{j+1} \not\equiv_{\ell+1} i_j + 1 \text{ or } i_{j-1} \not\equiv_{\ell+1} i_j - 1\right\}$$
$$\text{if char } K \neq 3$$
$$T(\Lambda)_4 := \left\{(\Lambda_{i_j^-, i_j^+})_{i_j^-, i_j^+} \mid m_{i_j} = 4\right\} \text{ if char } K \neq 2$$
$$T(\Lambda)_5 := \left\{(\Lambda_{i_j^-, i_j^+})_{i_p^-, i_p^+} \mid m_{i_j} = m_{i_p} = 2, i_p \not\equiv_{\ell+1} i_j \pm 1, j \neq p\right\}$$

Set

$$\mathcal{F}(\Lambda) := \{\beta_{\Lambda'} \mid \Lambda' \in \{\Lambda\} \cup F(\Lambda)_0 \cup F(\Lambda)_1\},$$
$$\mathcal{T}(\Lambda) := \{\beta_{\Lambda'} \mid \Lambda' \in \cup_{1\le j\le 5} T(\Lambda)_j\}.$$

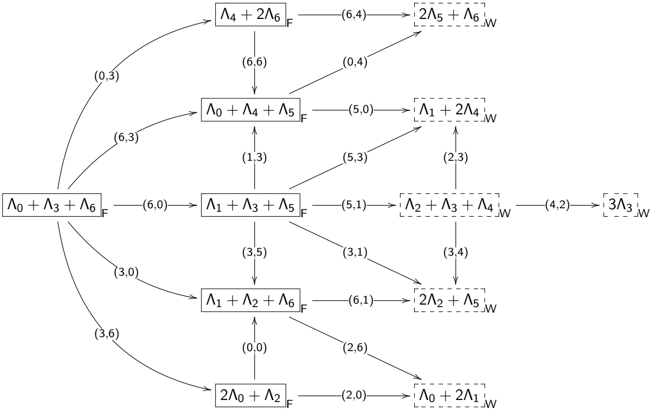
Theorem (Ariki-Song-W., 2023)

Suppose $|\Lambda| \geq 3$. Then, $R^\Lambda(\beta)$ is representation-finite if $\beta \in \mathcal{F}(\Lambda)$, tame if one of the following holds:

- $\beta = \delta$, $\Lambda = k\Lambda_i$, $\ell = 1$ with $t \neq \pm 2$,
- $\beta = \delta$, $\Lambda = k\Lambda_i$, $\ell \geq 2$ with $t \neq (-1)^{\ell+1}$,
- $\beta \in \mathcal{T}(\Lambda)$.

Otherwise, it is wild.

e.g., rep-type of $\vec{C}(\Lambda_0+\Lambda_3+\Lambda_6)$ in type $A_6^{(1)}$ is displayed as



e.g., rep-type of $\vec{C}(2\Lambda_2)$ in type $C_4^{(1)}$ is displayed as

$$
\begin{tikzcd}
2\Lambda_0{}_{\mathrm{W}} & & \\
2\Lambda_1{}_{\mathrm{W}} \arrow[u] \arrow[r] & \Lambda_0+\Lambda_2{}_{\mathrm{W}} \arrow[ul] & \\
\boxed{2\Lambda_2}_{\mathrm{F}} \arrow[u] \arrow[ur] \arrow[r] \arrow[d] \arrow[dr] & \boxed{\Lambda_1+\Lambda_3}_{\mathrm{F}} \arrow[ul] \arrow[u] \arrow[r] \arrow[dl] \arrow[d] & \Lambda_0+\Lambda_4{}_{\mathrm{T/W}} \arrow[ul] \arrow[dl] \\
2\Lambda_3{}_{\mathrm{W}} \arrow[r] \arrow[d] & \Lambda_2+\Lambda_4{}_{\mathrm{W}} \arrow[dl] & \\
2\Lambda_4{}_{\mathrm{W}} & &
\end{tikzcd}
$$

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Thank you! Any questions?

Background
- Categorification;
- Cyclotomic Hecke algebras;
- Bound quiver algebras;
- Representation type: rep-finite, tame, wild.

Objects
- Lie theoretic data;
- Cyclotomic KLR algebras;
- $\max^+(\Lambda)$ and $P_k^+(\Lambda)$;
- Rule to draw arrows;
- Rep-finite and tame sets in affine type A.