On brick finiteness of finite-dimensional algebras

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Outline

Introduction

Goal of Algebraic Representation Theory

Classify all indecomposable modules of a given algebra A and all morphisms between them, up to isomorphism.

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Quiver Representation Theory

Any (basic, connected) algebra A over an algebraically closed field K is isomorphic to a **bound quiver algebra** KQ/I.

An algebra A is said to be

- **rep-finite** if the number of indecomposable modules is finite.
- **tame** if A is not rep-finite, but all indecomposable modules can be organized in a one-parameter family in each dimension.
- **wild** if there exists a faithful exact K-linear functor from the module category of $K\langle x, y\rangle$ to $\operatorname{mod} A$.

Rep-finite path algebra

Gabriel's Theorem (Gabriel, 1972)

A path algebra $A = KQ$ is rep-finite if and only if the underlying graph of Q is one of Dynkin graphs:

- $\mathbb{A}_n$: $\circ - \circ - \circ - \cdots - \circ - \circ$
- $\mathbb{D}_n$:
$$\begin{tikzcd}[arrows=dash] \circ \arrow[dr] & & & & & \\ & \circ \arrow[r] & \circ \arrow[r] & \cdots \arrow[r] & \circ \arrow[r] & \circ \\ \circ \arrow[ur] & & & & & \end{tikzcd}$$
- $\mathbb{E}_6$:
$$\begin{tikzcd}[arrows=dash] & & \circ \arrow[d] & & \\ \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \end{tikzcd}$$
- $\mathbb{E}_7$:
$$\begin{tikzcd}[arrows=dash] & & \circ \arrow[d] & & & \\ \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \end{tikzcd}$$
- $\mathbb{E}_8$:
$$\begin{tikzcd}[arrows=dash] & & \circ \arrow[d] & & & & \\ \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \end{tikzcd}$$

Tame and Wild

Tame, e.g., $K(\circ \rightrightarrows \circ)$. Indecomposable modules:

$$\text{dim } 3: \quad K^2 \underset{(0,1)}{\overset{(1,0)}{\rightrightarrows}} K \qquad K \underset{(0,1)^t}{\overset{(1,0)^t}{\rightrightarrows}} K^2$$

$$\text{dim } 4: \quad K^2 \underset{J_2(0)}{\overset{I_2}{\rightrightarrows}} K^2 \qquad K^2 \underset{J_2(\lambda)}{\overset{I_2}{\rightrightarrows}} K^2$$

$$\vdots$$

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$$\text{dim } 4: \quad K^2 \overset{I_2}{\underset{J_2(0)}{\rightrightarrows}} K^2 \qquad K^2 \overset{I_2}{\underset{J_2(\lambda)}{\rightrightarrows}} K^2$$

$$\vdots$$

$$K^{n+1} \overset{[I_n,O]}{\underset{[O,I_n]}{\rightrightarrows}} K^n \qquad K^n \overset{I_n}{\underset{J_n(\lambda)}{\rightrightarrows}} K^n$$

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$$\vdots$$

$$K^{n+1} \underset{[O, I_n]}{\overset{[I_n, O]}{\rightrightarrows}} K^n \qquad K^n \underset{J_n(\lambda)}{\overset{I_n}{\rightrightarrows}} K^n$$

Wild, e.g., $K(\circ \rightrightarrows\!\!\!\rightarrow \circ)$. Indecomposable modules:

$$\text{dim } 3: \quad K^2 \begin{array}{c} \xrightarrow{(1,0)} \\ \xrightarrow{z} \\ \xrightarrow[(0,1)]{} \end{array} K \qquad z = (\lambda, \mu)$$

Trichotomy Theorem (Drozd, 1977)

The representation type of an algebra A (over K) is exactly one of rep-finite, tame and wild.

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It leads to two directions:

(1) Studying mod A in-depth, such as Auslander-Reiten theory, homological dimensions, triangulated categories, etc, for rep-finite and tame algebras;

(2) Studying nice subcategories of mod A, such as Serre subcategories, wide subcategories, etc, for wild algebras.

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Aim of this talk

To capture **some finite property** in wild cases.

Brick finiteness of algebras

A module M is called a **brick** if $\mathrm{End}_A(M) \simeq K$.

Then, A is said to be

(1) (Chindris-Kinser-Weyman, 2012) **Schur-representation-finite** if there are finitely many bricks of a fixed dimension.

(2) (Demonet-Iyama-Jasso, 2016) **brick-finite** if there are finitely many bricks in the module category of A.

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(2) (Demonet-Iyama-Jasso, 2016) **brick-finite** if there are finitely many bricks in the module category of A.

- (2) $\Rightarrow$ (1) is obvious.
- (1) $\Rightarrow$ (2) is not verified; no counterexample.

Wild, but brick-finite

Set $\Lambda_n = KQ/I_n$ with

$$Q: \begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[loop right, "\beta"] \end{tikzcd} \text{ and } I_n : \langle \beta^n, \alpha\beta^2 \rangle,\ n \geqslant 2,$$

the representation type of Λ_n is

- rep-finite if $n \leqslant 5$;
- tame if $n = 6$;
- wild if $n \geqslant 7$.

But, Λ_n admits only 4 bricks for any $n \geqslant 2$.

τ-tilting theory

τ-tilting theory was introduced by Adachi, Iyama and Reiten in 2014, as a completion to the classical tilting theory.

So far, τ-tilting theory is related to several different aspects in Representation Theory of Algebras:

- Categorical objects, such as torsion classes, silting complexes;
- Combinatorial objects, such as bricks, semibricks;
- Lattice theory, such as the lattice of torsion classes;
- Geometric objects, such as the modern Brauer-Thrall conjecture, wall-and-chamber structures.

Auslander-Reiten translation

Nakayama functor $\nu = D(-)^* : \text{proj } A \to \text{inj } A$

- $D(-) = \text{Hom}_K(-, K) : \text{mod } A \longleftrightarrow \text{mod } A^{\text{op}}$
- $(-)^* = \text{Hom}_A(-, A) : \text{proj } A \longleftrightarrow \text{proj } A^{\text{op}}$

Let M be an A-module with a minimal projective presentation

$$P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0,$$

the **Auslander-Reiten translation** τM is defined by the following exact sequence

$$0 \longrightarrow \tau M \longrightarrow \nu P_1 \xrightarrow{\nu f_1} \nu P_0,$$

that is, $\tau M = \ker \nu f_1$.

Definition 2.1 (Adachi-Iyama-Reiten, 2014)

Let M be a right A-module. Then,

(1) M is called τ-rigid if $\mathrm{Hom}_A(M, \tau M) = 0$.

(2) M is called τ-tilting if M is τ-rigid and $|M| = |A|$.

(3) M is called support τ-tilting if M is a τ-tilting (A/AeA)-module for an idempotent e of A.

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We define the sets $\mathrm{i}\tau\text{-rigid}\,A$, $\tau\text{-tilt}\,A$, $\mathrm{s}\tau\text{-tilt}\,A$, respectively. Then,

$$\mathrm{i}\tau\text{-rigid}\,A \subseteq \tau\text{-tilt}\,A \subseteq \mathrm{s}\tau\text{-tilt}\,A$$

Mutation

Reminder: $M_1 \oplus \cdots \oplus M_j \oplus \cdots \oplus M_n \Rightarrow M_1 \oplus \cdots \oplus M_j^* \oplus \cdots \oplus M_n$.

- add(M): the full subcategory whose objects are direct summands of finite direct sums of copies of M;
- Fac(M): the full subcategory whose objects are factor modules of finite direct sums of copies of M.

Definition 2.2 (AIR, 2014)

Let $M = M_1 \oplus \cdots \oplus M_j \oplus \cdots \oplus M_n$ with $M_j \notin \mathrm{Fac}(M/M_j)$. Take a minimal left add(M/M_j)-approximation π with an exact sequence

$$M_j \xrightarrow{\pi} Z \longrightarrow \mathrm{coker}\ \pi \longrightarrow 0.$$

We call $\mu_j^-(M) := \mathrm{coker}\ \pi \oplus (M/M_j)$ the left mutation of M with respect to M_j, which is again a support τ-tilting A-module.

Mutation Graph

We draw an arrow $M \to \mu_j^-(M)$, it gives a graph $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\, A)$.

For example, $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\, \Lambda_2)$ is displayed as

$$\begin{tikzcd}
\begin{smallmatrix}1\\2\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\2\end{smallmatrix} \arrow[rrr] \arrow[d] & & & \begin{smallmatrix}2\\2\end{smallmatrix} \arrow[d] \\
\begin{smallmatrix}1\\2\\2\end{smallmatrix} \oplus \begin{smallmatrix} & & 1\\ 1 & & 2\\ & 2 & \end{smallmatrix} \arrow[r] & 1 \oplus \begin{smallmatrix} & & 1\\ 1 & & 2\\ & 2 & \end{smallmatrix} \arrow[r] & 1 \arrow[r] & 0
\end{tikzcd}.$$

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\begin{smallmatrix}1\\2\\2\end{smallmatrix} \oplus \begin{smallmatrix} & & 1\\ 1 & & 2\\ & 2 & \end{smallmatrix} \arrow[r] & 1 \oplus \begin{smallmatrix} & & 1\\ 1 & & 2\\ & 2 & \end{smallmatrix} \arrow[r] & 1 \arrow[r] & 0
\end{tikzcd}.$$

Proposition 2.3 (AIR, 2014)

If the mutation graph $\mathcal{H}(\mathsf{s}\tau\text{-tilt}\, A)$ contains a finite connected component Δ, then $\mathcal{H}(\mathsf{s}\tau\text{-tilt}\, A) = \Delta$.

Connection with brick finiteness

- brick A: the set of bricks in $\operatorname{mod} A$
- fbrick A: the set of bricks M such that the smallest torsion class $\mathsf{T}(M)$ containing M is functorially finite.

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Theorem 2.4 (Demonet-Iyama-Jasso, 2016)

There exists a bijection between iτ-rigid A and fbrick A given by

$$X \mapsto X/\mathrm{rad}_B(X),$$

where $B := \mathrm{End}_A(X)$. If iτ-rigid A is finite, brick A = fbrick A.

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e.g.,

iτ-rigid Λ_2	$\begin{smallmatrix} 1 \\ 2 \\ 2 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}$	$\begin{smallmatrix} & 1 \\ 1 & 2 \\ 2 & \end{smallmatrix}$	1
brick Λ_2	$\begin{smallmatrix} 1 \\ 2 \\ 2 \end{smallmatrix}$	2	$\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$	1

Known Result

The brick finiteness is known, for example, for

- preprojective algebras of Dynkin type (Mizuno, 2014);
- algebras with radical square zero (Adachi, 2016);
- cycle-finite algebras (Malicki-Skowroński, 2016);
- Brauer graph algebras (Adachi-Aihara-Chan, 2018);
- gentle algebras (Plamondon, 2018);
- (special) biserial algebras (Mousavand, 2019; Schroll-Treffinger-Valdivieso, 2021);
- cluster-tilted algebras (Zito, 2019);
- minimal wild two-point algebras (W., 2019).
- tensor product algebras (Miyamoto-W., 2019);
- quasi-tilted algebras, locally hereditary algebras, etc., (Aihara-Honma-Miyamoto-W., 2020).

Reduction Theorem

Proposition 2.5 (Demonet-Iyama-Jasso, 2016)

If A is brick-finite, then

(1) A/I is brick-finite, for any two-sided ideal I of A.

(2) eAe is brick-finite, for any idempotent e of A.

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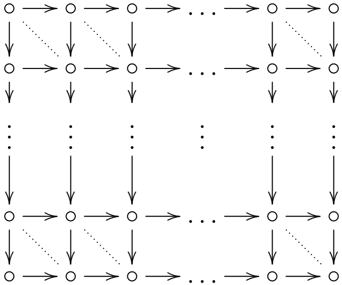
Proposition 2.6 (Eisele-Janssens-Raedschelders, 2018)

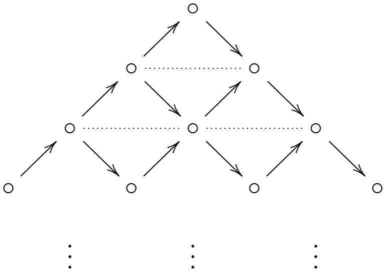
Let I be a two-sided ideal generated by central elements which are contained in the radical of A. Then, there exists a poset isomorphism between $\mathsf{s}\tau\text{-tilt}\,A$ and $\mathsf{s}\tau\text{-tilt}\,(A/I)$.

Upper and lower boundary

Upper boundary

Let A be an algebra without loops and oriented cycles. We want to see what happens if A has lots of vertices. For example,





This motivates us to consider simply connected algebras.

Simply connected algebra

Let $A = KQ/I$ without loops and oriented cycles. We consider the fundamental group $\Pi_1(Q, I)$ of A. Then, A is said to be a **simply connected algebra** if, for every bound quiver presentation KQ/I of A, $\Pi_1(Q, I)$ is trivial. (Assem-Skowroński, 1988)

We have the following examples.

(1) All tree algebras are simply connected.

(2) A path algebra KQ is simply connected if and only if Q is a tree. For example, KQ is not simply connected if

$$Q = \begin{tikzcd} \circ \arrow[r] \arrow[d] & \circ \arrow[d] \\ \circ \arrow[r] & \circ \end{tikzcd}.$$

Theorem 3.1 (W., 2019)

Let A be a simply connected algebra. Then,

$$A \text{ is brick-finite} \Leftrightarrow A \text{ is rep-finite.}$$

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Sketch of the proof:

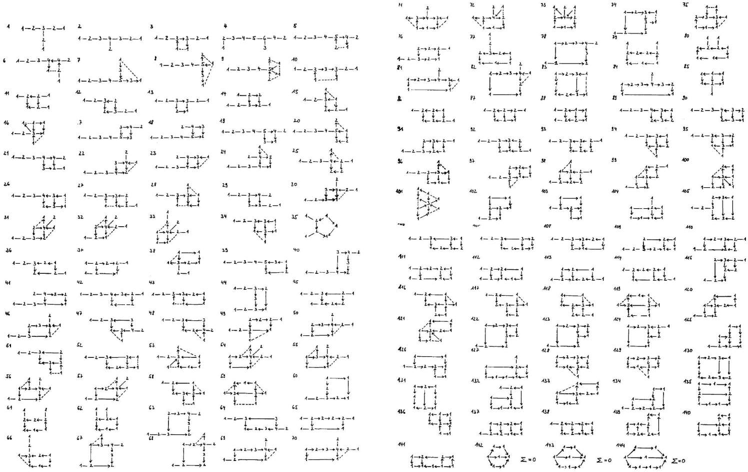
- A: rep-finite $\Rightarrow$ τ-tilting finite, obvious;
- A: rep-infinite

  $\Rightarrow$ there exists an idempotent e of A such that eAe is one of concealed algebras of type $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$ (Bongartz, 1984);

  $\Rightarrow$ eAe is brick-infinite;

  $\Rightarrow$ A is brick-infinite (Proposition 2.4).

A complete list of concealed algebras of type $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$ is given as follows.



Rectangle Quiver

Let $B_{m,n}$ ($m \leqslant n$) be the algebra given by the following quiver with all possible commutativity relations:

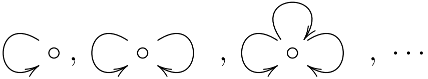
$$
\begin{tikzcd}
(1,1) \arrow[r] \arrow[d] \arrow[dr, dotted, no head] & (1,2) \arrow[r] \arrow[d] \arrow[dr, dotted, no head] & \circ \arrow[r] \arrow[d] & \dots \arrow[r] & \circ \arrow[r] \arrow[d] \arrow[dr, dotted, no head] & (1,n) \arrow[d] \\
(2,1) \arrow[r] \arrow[d] & (2,2) \arrow[r] \arrow[d] & \circ \arrow[r] \arrow[d] & \dots \arrow[r] & \circ \arrow[r] \arrow[d] & \circ \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots & \vdots \arrow[d] & \vdots \arrow[d] \\
\circ \arrow[r] \arrow[d] \arrow[dr, dotted, no head] & \circ \arrow[r] \arrow[d] \arrow[dr, dotted, no head] & \circ \arrow[r] \arrow[d] & \dots \arrow[r] & \circ \arrow[r] \arrow[d] \arrow[dr, dotted, no head] & \circ \arrow[d] \\
(m,1) \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \dots \arrow[r] & \circ \arrow[r] & (m,n)
\end{tikzcd}
$$

Then, $B_{m,n}$ is brick-finite if and only if

$$(m,n) \in \{(1,n),(2,2),(2,3),(2,4)\}.$$

Lower boundary

A local algebra is always brick-finite, whose quiver is given as



Lower boundary

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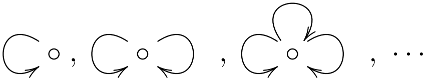
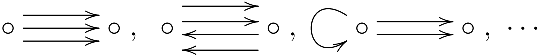
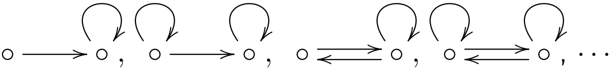


This forces us to focus on $A = KQ/I$ with only two vertices:



or



Two-point algebra

Proposition 3.2

The Kronecker algebra $K(\ 1 \rightrightarrows 2\ )$ is brick-infinite.

Proof: It is well-known that $K \underset{1}{\overset{\lambda}{\rightrightarrows}} K$ is a brick, for any $\lambda \in K$.

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<u>Proof:</u> It is well-known that $K \underset{1}{\overset{\lambda}{\rightrightarrows}} K$ is a brick, for any $\lambda \in K$.

We only need to consider

$$Q(m,n) := \begin{tikzcd} \vdots \, 1 \arrow[loop above, "\alpha_1"] \arrow[loop below, "\alpha_m"'] \arrow[r, shift left, "\mu"] & 2 \, \vdots \arrow[l, shift left, "\nu"] \arrow[loop above, "\beta_1"] \arrow[loop below, "\beta_n"'] \end{tikzcd}$$

Theorem 3.3 (W., 2022)

Let $A = KQ(m, n)/I$ be a monomail algebra with $\operatorname{rad}^3 A = 0$. Then, A is brick-finite if and only if it does not have $\Delta = KQ/I$:

$$Q: \begin{tikzcd} 1 \arrow[r] & 2 \arrow[loop above, "\beta_1"] \arrow[loop below, "\beta_2"] \end{tikzcd} \quad \text{and } I : \langle \beta_1^2, \beta_2^2, \beta_1\beta_2, \beta_2\beta_1 \rangle,$$

or its opposite algebra as a quotient algebra.

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Sketch of the proof:

(1) $\mathsf{s}\tau\text{-tilt}\, A \simeq \mathsf{s}\tau\text{-tilt}\,(A/J)$, $J \subseteq \operatorname{rad} A \cap Z(A)$;

(2) Δ is brick-infinite, using silting theory.

Silting Theory

Proposition 3.4 (AIR, 2014)

There exists a poset isomorphism between sτ-tilt A and 2-silt A, the bijection $\mathcal{F}$ is given by

$$M \longmapsto (P_1 \oplus P \xrightarrow{\binom{f}{0}} P_0)\ ,$$

where (M, P) is the support τ-tilting pair corresponding to M and $P_1 \xrightarrow{f} P_0 \to M \to 0$ is the minimal projective presentation of M.

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A mutation chain: $M^{(1)} \to M^{(2)} \to \cdots \to M^{(k)} \to \cdots$

$$\begin{tikzcd}
\mathcal{F}(M^{(1)}) \arrow[r] \arrow[d, dotted, "\mathrm{End}"] & \mathcal{F}(M^{(2)}) \arrow[r] \arrow[d, dotted, "\mathrm{End}"] & \cdots \arrow[r] & \mathcal{F}(M^{(2k-1)}) \arrow[r] \arrow[d, dotted, "\mathrm{End}"] & \mathcal{F}(M^{(2k)}) \arrow[r] \arrow[d, dotted, "\mathrm{End}"] & \cdots \\
B & B^{\mathrm{op}} & \cdots & B & B^{\mathrm{op}} & \cdots
\end{tikzcd}$$

Proposition 3.4 (W., 2022)

Let $A = KQ(1,1)/I$ be a monomail algebra with $\operatorname{rad}^5 A = 0$. Then, A is brick-finite if and only if it does not have one of

- $\begin{tikzcd} \circ \arrow[r, "\mu"] & \circ \arrow[loop right, "\beta"] \end{tikzcd}$ with $\beta^4 = 0$,
- $\begin{tikzcd} \circ \arrow[r, shift left, "\mu"] & \circ \arrow[l, shift left, "\nu"] \arrow[loop right, "\beta"] \end{tikzcd}$ with $\beta^3 = \beta\nu = \nu\mu\nu = \nu\mu\beta^2 = 0$,
- $\begin{tikzcd} \circ \arrow[loop left, "\alpha"] \arrow[r, "\mu"] & \circ \arrow[loop right, "\beta"] \end{tikzcd}$ with $\alpha^2 = \beta^2 = 0$,

and their opposite algebras as a quotient algebra.

Application

Derived Equivalence Class

- A is derived equivalent to $B \Leftrightarrow \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, A) \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, B)$

Theorem 4.1 (Ariki-Song-W., 2024)

Let $A_1, A_2, \ldots, A_s$ be pairwise derived equivalent symmetric algebras. Suppose the following conditions hold.

(1) A_i is brick-finite, for all $1 \le i \le s$.

(2) $\mathrm{End}(\mu_k^-(A_i)) \in \{A_1, A_2, \ldots, A_s\}$, for any k and all $1 \le i \le s$.

Then, any algebra B which has derived equivalence

$$\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, B) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, A_1)$$

is included in $\{A_1, A_2, \ldots, A_s\}$.

Derived Equivalence Class

We consider the following quiver:

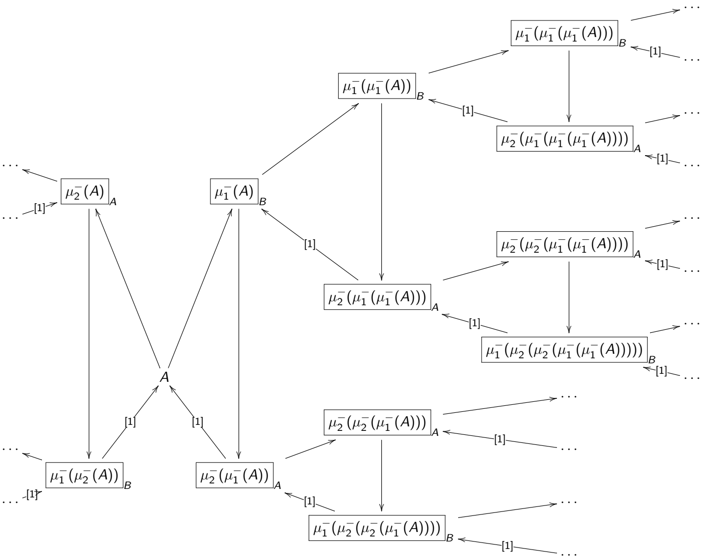
$$Q: \begin{tikzcd} \circ \arrow[loop left, "\alpha"] \arrow[r, shift left, "\mu"] & \circ \arrow[l, shift left, "\nu"] \arrow[loop right, "\beta"] \end{tikzcd},$$

and define

- $A := KQ/\langle \alpha^2, \beta^2 - \nu\mu, \alpha\mu - \mu\beta, \beta\nu - \nu\alpha \rangle$.
- $B := KQ/\langle \alpha^2 - \mu\nu, \beta^2 - \nu\mu, \alpha\mu - \mu\beta, \beta\nu - \nu\alpha, \mu\nu\mu, \nu\mu\nu \rangle$.

Proposition 4.2

If C is derived equivalent to A, then C is isomorphic to A or B.



A
$\mu_2^-(A)_A$
$\mu_1^-(A)_B$
$\mu_1^-(\mu_2^-(A))_B$
$\mu_2^-(\mu_1^-(A))_A$
$\mu_1^-(\mu_1^-(A))_B$
$\mu_2^-(\mu_1^-(\mu_1^-(A)))_A$
$\mu_2^-(\mu_2^-(\mu_1^-(A)))_A$
$\mu_1^-(\mu_2^-(\mu_2^-(\mu_1^-(A))))_B$
$\mu_1^-(\mu_1^-(\mu_1^-(A)))_B$
$\mu_2^-(\mu_1^-(\mu_1^-(\mu_1^-(A))))_A$
$\mu_2^-(\mu_2^-(\mu_1^-(\mu_1^-(A))))_A$
$\mu_1^-(\mu_2^-(\mu_2^-(\mu_1^-(\mu_1^-(A)))))_B$
[1]
. . .

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Thank you! Any questions?

$$\begin{cases} \text{Quiver representation theory;} \\ \text{Representation type: rep-finite, tame, wild;} \\ \text{Brick finiteness of algebras;} \\ \tau\text{-tilting theory;} \end{cases}$$

$$\begin{cases} \text{Simply connected algebras;} \\ \text{Two-point algebras;} \\ \text{Silting theory;} \\ \text{Derived equivalence class.} \end{cases}$$