Representation type of cyclotomic quiver Hecke algebras[1]

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[1]Collaborations with Susumu Ariki, Berta Hudak, and Linliang Song.

Outline

Introduction

Hecke algebra of type A

The **symmetric group** $\mathfrak{S}_n$ (= permutation group of $\{1, 2, \cdots, n\}$) is generated by $\{s_i = (i, i+1) \mid 1 \leq i \leq n-1\}$ subject to

$$s_i^2 = 1, (\Leftrightarrow (s_i + 1)(s_i - 1) = 0)$$

$$s_i s_j = s_j s_i \text{ if } |i-j| \neq 1, \quad s_i s_j s_i = s_j s_i s_j \text{ if } |i-j| = 1.$$

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The **Iwahori-Hecke algebra** $\mathcal{H}(\mathfrak{S}_n)$ is the $\mathbb{Z}[q, q^{-1}]$-algebra generated by $\{T_i \mid 1 \le i \le n-1\}$ subject to

$$T_i^2 = (q-1)T_i + q, (\Leftrightarrow (T_i + 1)(T_i - q) = 0)$$

$$T_i T_j = T_j T_i \text{ if } |i - j| \neq 1, \quad T_i T_j T_i = T_j T_i T_j \text{ if } |i - j| = 1.$$

From the perspective of Lie theory, one wants to know

- the irreducible representations of $\mathcal{H}(\mathfrak{S}_n)$,
- the decomposition numbers of $\mathcal{H}(\mathfrak{S}_n)$.

This is accompanied by the rise of many theories, such as categorification theory, cellular algebra theory, crystal bases theory, Kazhdan-Lusztig theory, Lascoux-Leclerc-Thibon algorithm, etc.

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Now, we have many generalizations of $\mathcal{H}(\mathfrak{S}_n)$,

- Hecke algebras of Coxeter groups, i.e., of type A, B, D, etc.
- Cyclotomic Hecke algebras (a.k.a. Ariki-Koike algebras). See [Ariki-Koike, 1994], [Broue-Malle, 1993], and [Cherednik 1987].
- Cyclotomic quiver Hecke algebras (a.k.a. Cyclotomic KLR algebras). See [Khovanov-Lauda, 2009] and [Rouquier, 2008].

The representation type is completely determined for many classes of algebras, such as

(1) Hecke alg's in type A, B (Erdmann-Nakano 2001, Ariki-Mathas 2004);

(2) Cyclotomic quiver Hecke alg's of level 1 in affine type A, C, D (Ariki-Iijima-Park 2014, 2015); of level 2 in affine type A (Ariki 2017);

(3) Schur/q-Schur/Borel-Schur/infinitesimal-Schur alg's (Xi 1993, Erdmann 1993, Doty-Erdmann-Martin 1999, Erdmann-Nakano 2001, etc);

(4) block alg's of category $\mathcal{O}$; (Futorny-Nakano-Pollack 1999, Boe-Nakano 2005, etc)

Preview in affine type A

Main Theorem (Ariki-Song-W., 2023)

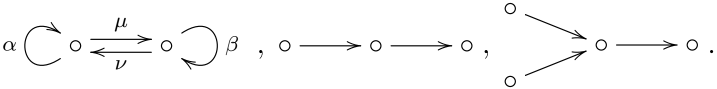
Suppose $|\Lambda| \geq 3$. The cyclotomic quiver Hecke algebra $R^\Lambda(\beta)$ of type $A_\ell^{(1)}$ is rep-finite if $\beta \in \mathcal{F}(\Lambda)$, tame if one of the following holds:

- $\beta = \delta$, $\Lambda = k\Lambda_i$, $\ell = 1$ with $t \neq \pm 2$,
- $\beta = \delta$, $\Lambda = k\Lambda_i$, $\ell \geq 2$ with $t \neq (-1)^{\ell+1}$,
- $\beta \in \mathcal{T}(\Lambda)$.

Otherwise, $R^\Lambda(\beta)$ is wild.

Quiver Representation Theory

Quivers:



- paths: e.g., $(\alpha\mu\beta\nu)^m$, $(\mu\nu)^n\alpha^k$, $(\alpha\mu\nu)^k(\mu\beta\nu)^m$, ...

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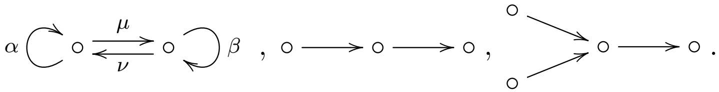

- paths: e.g., $(\alpha\mu\beta\nu)^m$, $(\mu\nu)^n\alpha^k$, $(\alpha\mu\nu)^k(\mu\beta\nu)^m$, ...

Bound quiver algebra $A = KQ/I$:

$$I = \langle \sum \lambda_i \omega_i, \cdots \rangle$$

- $\lambda_i \in K$ and ω_i is a path but not an arrow.

Representation type of algebra

Theorem (Drozd, 1977)

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- **rep-finite** if the number of indecomposable modules is finite.
- **tame** if it is not rep-finite, but all indecomposable modules can be organized in a one-parameter family in each dimension.

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"The representation type of symmetric algebras is preserved under derived equivalence." *(Rickard 1991, Krause 1998)*

Some examples related to Hecke algebras.

- rep-finite: e.g., Brauer tree algebras

$$1 \rightleftarrows 2 \rightleftarrows 3 \rightleftarrows 4 \quad \Rightarrow \quad \begin{matrix}1\\2\\1\end{matrix} \oplus \begin{matrix} & 2 & \\ 1 & & 3 \\ & 2 & \end{matrix} \oplus \begin{matrix} & 3 & \\ 2 & & 4 \\ & 3 & \end{matrix} \oplus \begin{matrix}4\\3\\4\end{matrix}$$

- tame: e.g., Brauer graph algebras
- wild:

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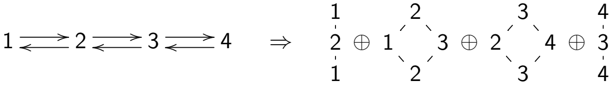

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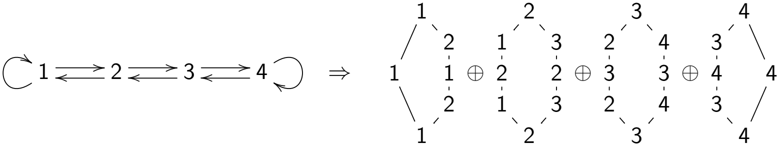

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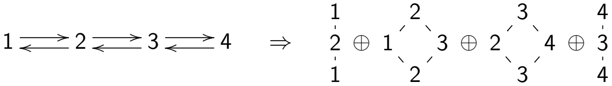

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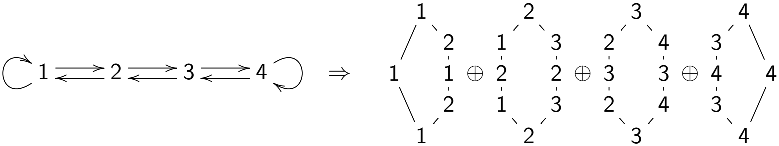

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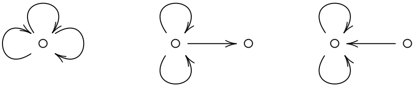


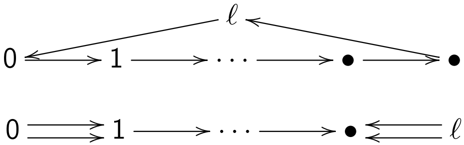
Cyclotomic quiver Hecke algebras

Lie theoretic data

Let $I = \{0, 1, ..., \ell\}$ be an index set. Recall that

$A_\ell^{(1)}$:

$C_\ell^{(1)}$:



$+B_\ell^{(1)}, D_\ell^{(1)}, A_{2\ell}^{(2)}, A_{2\ell-1}^{(2)}, D_{\ell+1}^{(2)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, F_4^{(1)}, G_2^{(1)}, E_6^{(2)}, D_4^{(3)}.$

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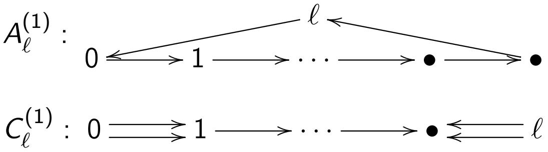

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Set $n_{ij} := \#(i \to j)$.

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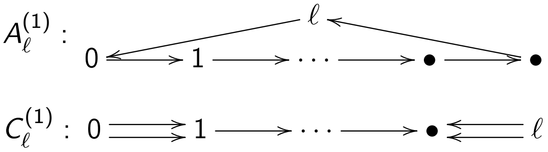

$$+B_\ell^{(1)}, D_\ell^{(1)}, A_{2\ell}^{(2)}, A_{2\ell-1}^{(2)}, D_{\ell+1}^{(2)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, F_4^{(1)}, G_2^{(1)}, E_6^{(2)}, D_4^{(3)}.$$

Set $n_{ij} := \#(i \to j)$. We define the **Cartan matrix** $\mathrm{A} = (a_{ij})_{i,j \in I}$ by

$$a_{ii} = 2, \quad a_{ij} = \begin{cases} -n_{ij} & \text{if } n_{ij} > n_{ji}, \\ -1 & \text{if } n_{ij} < n_{ji}, \\ -n_{ij} - n_{ji} & \text{otherwise}, \end{cases} \qquad (i \neq j).$$

Let $(\mathrm{A}, P, \Pi, P^\vee, \Pi^\vee)$ be the **Cartan datum** in type $X^{(1)}$, where

- $P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_\ell \oplus \mathbb{Z}\delta$ is the weight lattice;
- $\Pi = \{\alpha_i \mid 0 \le i \le \ell\} \subset P$ is the set of simple roots;
- $P^\vee = \mathrm{Hom}(P, \mathbb{Z})$ is the coweight lattice;
- $\Pi^\vee = \{h_i \mid 0 \le i \le \ell\} \subset P^\vee$ is the set of simple coroots.

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We have

$$\langle h_i, \alpha_j \rangle = a_{ij}, \quad \langle h_i, \Lambda_j \rangle = \delta_{ij} \quad \text{for } 0 \le i, j \le \ell.$$

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The null root is δ, e.g.,

$$\delta = \begin{cases} \alpha_0 + \alpha_1 + \cdots + \alpha_\ell & \text{if } X = A_\ell, \\ \alpha_0 + 2(\alpha_1 + \cdots + \alpha_{\ell-1}) + \alpha_\ell & \text{if } X = C_\ell. \end{cases}$$

Quiver Hecke algebra

The **quiver Hecke algebra** $R(n)$ associated with $(Q_{i,j}(u,v))_{i,j\in I}$ is the K-algebra generated by

$$\{e(\nu) \mid \nu = (\nu_1, \nu_2, \ldots, \nu_n) \in I^n\}, \quad \{x_i \mid 1 \le i \le n\}, \quad \{\psi_j \mid 1 \le j \le n-1\},$$

subject to the following relations:

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subject to the following relations:

(1) $e(\nu)e(\nu')=\delta_{\nu,\nu'}e(\nu),\ \sum_{\nu\in I^n}e(\nu)=1,\ x_ix_j=x_jx_i,\ x_ie(\nu)=e(\nu)x_i.$

(2) $\psi_ie(\nu)=e(s_i(\nu))\psi_i,\ \psi_i\psi_j=\psi_j\psi_i$ if $|i-j|>1$.

(3) $\psi_i^2e(\nu)=Q_{\nu_i,\nu_{i+1}}(x_i,x_{i+1})e(\nu)$.

(4) $(\psi_ix_j-x_{s_i(j)}\psi_i)e(\nu)=\begin{cases}-e(\nu) & \text{if } j=i \text{ and } \nu_i=\nu_{i+1},\\ e(\nu) & \text{if } j=i+1 \text{ and } \nu_i=\nu_{i+1},\\ 0 & \text{otherwise.}\end{cases}$

(5) $(\psi_{i+1}\psi_i\psi_{i+1}-\psi_i\psi_{i+1}\psi_i)e(\nu)=\begin{cases}\frac{Q_{\nu_i,\nu_{i+1}}(x_i,x_{i+1})-Q_{\nu_i,\nu_{i+1}}(x_{i+2},x_{i+1})}{x_i-x_{i+2}}e(\nu) & \text{if } \nu_i=\nu_{i+2},\\ 0 & \text{otherwise.}\end{cases}$

A family of polynomials in affine type A

Fix $t \in K$ if $\ell = 1$ and $0 \neq t \in K$ if $\ell \geq 2$.

For $i, j \in I$, we take $Q_{i,j}(u, v) \in K[u, v]$ such that $Q_{i,i}(u, v) = 0$, $Q_{i,j}(u, v) = Q_{j,i}(v, u)$ and if $\ell \geq 2$,

$$\begin{aligned} Q_{i,i+1}(u, v) &= u + v \text{ if } 0 \leq i < \ell, \\ Q_{\ell,0}(u, v) &= u + tv, \\ Q_{i,j}(u, v) &= 1 \text{ if } j \not\equiv_{\ell+1} i, i \pm 1. \end{aligned}$$

If $\ell = 1$, we take $Q_{0,1}(u, v) = u^2 + tuv + v^2$.

Cyclotomic quiver Hecke algebras

Set

$$\Lambda = a_0\Lambda_0 + a_1\Lambda_1 + \cdots + a_\ell\Lambda_\ell, \; a_i \in \mathbb{Z}_{\geq 0}.$$

The **cyclotomic quiver Hecke algebra** $R^\Lambda(n)$ is defined as the quotient of $R(n)$ modulo the relation

$$x_1^{\langle h_{\nu_1}, \Lambda \rangle} e(\nu) = 0.$$

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Set

$$\beta = b_0\alpha_0 + b_1\alpha_1 + \cdots + b_\ell\alpha_\ell,\ \ b_i \in \mathbb{Z}_{\geq 0},$$

with $|\beta| = b_1 + \cdots + b_\ell = n$, we define

$$R^\Lambda(\beta) := e(\beta) R^\Lambda(n) e(\beta),$$

where $e(\beta) := \sum_{\nu \in I^\beta} e(\nu)$ with $I^\beta = \left\{ \nu = (\nu_1, \nu_2, \ldots, \nu_n) \in I^n \mid \sum_{i=1}^{n} \alpha_{\nu_i} = \beta \right\}$.

An example

Set $\Lambda = k\Lambda_0, \ell = 2$. Then, $I = \{0, 1, 2\}$ and $R(3)$ is generated by

$$\{e(000), \cdots, e(012), \cdots, e(212), \cdots\}, \{x_1, x_2, x_3\}, \{\psi_1, \psi_2\},$$

subject to the relations.

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subject to the relations.

Set $\beta = \alpha_0 + \alpha_1 + \alpha_2$. Then, $R^\Lambda(\beta)$ is generated by

$$\{e(012), e(021), e(102), e(120), e(201), e(210)\}, \{x_1, x_2, x_3\}, \{\psi_1, \psi_2\},$$

subject to

- $e(102) = e(120) = e(201) = e(210) = 0$, $x_1^k e(012) = x_1^k e(021) = 0$;
- $\psi_1 e(012) = \psi_1 e(021) = 0$, $\psi_2 e(012) = e(021)\psi_2$;
- $x_2 e(012) = -x_1 e(012)$, $x_2 e(021) = -t x_1 e(021)$;
- $x_3^2 e(012) = t x_1^2 e(012) + (1-t) x_1 x_3 e(012)$, etc.

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- $R^\Lambda(\beta) \sim_{\text{derived}} R^\Lambda(\beta')$ if both $\Lambda - \beta$ and $\Lambda - \beta'$ lie in

$$\{\mu - m\delta \mid \mu \in \max^+(\Lambda), m \in \mathbb{Z}_{\geq 0}\},$$

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  which is the W-orbit of the set $P(\Lambda)$ of weights of $V(\Lambda)$, where W is the affine symmetric group and $V(\Lambda)$ is the integrable highest weight module of the quantum group. See: [Chuang-Rouquier, 2008].
- A weight $\mu \in P(\Lambda)$ is maximal if $\mu + \delta \notin P(\Lambda)$. We define

$$\max^+(\Lambda) := \{\mu \in P^+ \mid \mu \text{ is maximal}\},$$

  where $P^+ := \{\mu \in P \mid \langle h_i, \mu\rangle \in \mathbb{Z}_{\geq 0}, i \in I\}$.

$\max^+(\Lambda)$

Theorem (Kim-Oh-Oh, 2020)

There is a bijection $\phi_\Lambda : \max^+(\Lambda) \to P_k^+(\Lambda)$.

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$$|\Lambda| := a_{i_1} + \cdots + a_{i_j} \quad \text{and} \quad \mathrm{ev}(\Lambda) := i_1 + \cdots + i_n.$$

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In type $A_\ell^{(1)}$, we have

$$P_k^+(\Lambda) := \left\{ \Lambda' \in P^+ \mid |\Lambda| = |\Lambda'| = k, \mathrm{ev}(\Lambda) \equiv_{\ell+1} \mathrm{ev}(\Lambda') \right\}.$$

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In type $A^{(1)}_\ell$, we have

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In type $C^{(1)}_\ell$, we have

$$P^+_k(\Lambda) := \left\{\Lambda' \in P^+ \mid |\Lambda| = |\Lambda'| = k, \mathrm{ev}(\Lambda) \equiv_2 \mathrm{ev}(\Lambda') \right\}.$$

Recall that $\langle h_i, \Lambda_j \rangle = \delta_{ij}$. We define $y_i := \langle h_i, \Lambda - \Lambda' \rangle$ and

$$Y_{\Lambda'} := (y_0, y_1, \ldots, y_\ell) \in \mathbb{Z}^{\ell+1}.$$

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Theorem (Ariki-Song-W., 2023)

The bijection $\phi_\Lambda^{-1} : P_k^+(\Lambda) \to \max^+(\Lambda)$ is given by

$$\Lambda' \mapsto \Lambda - \sum_{i=0}^{\ell} x_i \alpha_i,$$

where $X = (x_0, x_1, \dots, x_\ell)$ is the unique solution of $\mathsf{A}X^t = Y^t_{\Lambda'}$ satisfying

$$x_i \geq 0 \quad \text{and} \quad \min\{x_i - \delta\} < 0.$$

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$$x_i \geq 0 \quad \text{and} \quad \min\{x_i - \delta\} < 0.$$

We denote $\beta_{\Lambda'} := \sum_{i=0}^{\ell} x_i \alpha_i$.

Proof strategy in affine type A

$$\Lambda - \beta \in \{\mu - m\delta \mid \mu \in \max^+(\Lambda), m \in \mathbb{Z}_{\geq 0}\}$$

$$\Leftrightarrow \Lambda - \beta \in \{\Lambda - \beta_{\Lambda'} - m\delta \mid \Lambda' \in P_k^+(\Lambda), m \in \mathbb{Z}_{\geq 0}\}.$$

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$$\Leftrightarrow \Lambda - \beta \in \{\Lambda - \beta_{\Lambda'} - m\delta \mid \Lambda' \in P_k^{+}(\Lambda), m \in \mathbb{Z}_{\geq 0}\}.$$

Step 1: We show that $R^{\Lambda}(\beta_{\Lambda'} + m\delta)$ is wild for all $m \geq 1$ if $\beta_{\Lambda'} \neq 0$ and $R^{\Lambda}(m\delta)$ is wild for all $m \geq 2$, by using some **new reduction theorems**.

(If $R^{\Lambda}(\gamma)$ is not wild, we set $\gamma \in \mathcal{NW}(\Lambda) \cup \{\delta\}$.)

Step 2: We determine the representation type of $R^\Lambda(\gamma)$ for $\gamma \in \mathcal{T}(\Lambda) \cup \{\delta\}$, via case-by-case consideration.

(A systematic approach developed by Ariki and his collaborators is well applied to find the quiver presentation of $R^\Lambda(\gamma)$.)

Step 2: We determine the representation type of $R^{\Lambda}(\gamma)$ for $\gamma \in \mathcal{T}(\Lambda) \cup \{\delta\}$, via case-by-case consideration.

(A systematic approach developed by Ariki and his collaborators is well applied to find the quiver presentation of $R^{\Lambda}(\gamma)$.)

Step 3: We show that

$$\mathcal{NW}(\Lambda) \subset \mathcal{T}(\Lambda)$$

via case-by-case consideration on small k (i.e., $k = 3, 4, 5, 6$) and via induction on $k \geq 7$.

Structure of $P_k^+(\Lambda)$

Constructions in affine type A

Recall that, in type $A_\ell^{(1)}$,

$$P_k^+(\Lambda) = \left\{ \Lambda' \in P^+ \mid |\Lambda| = |\Lambda'| = k, \mathrm{ev}(\Lambda) \equiv_{\ell+1} \mathrm{ev}(\Lambda') \right\}.$$

Set $\Lambda' = \Lambda_i + \Lambda_j + \tilde{\Lambda} \in P_k^+(\Lambda)$ with $j \not\equiv_{\ell+1} i - 1$. We have

$$\Lambda'_{i^-, j^+} := \Lambda_{i-1} + \Lambda_{j+1} + \tilde{\Lambda} \in P_k^+(\Lambda).$$

Constructions in affine type A

Recall that, in type $A_\ell^{(1)}$,

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Set $\Lambda' = \Lambda_i + \Lambda_j + \tilde{\Lambda} \in P_k^+(\Lambda)$ with $j \not\equiv_{\ell+1} i - 1$. We have

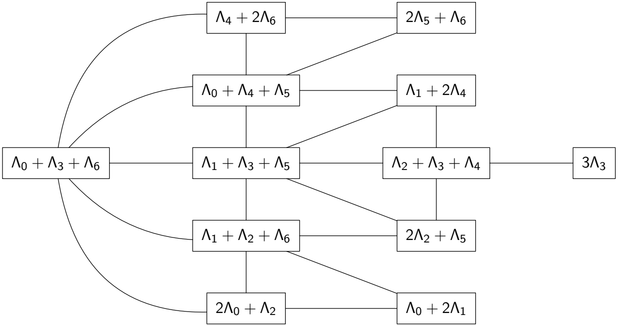
$$\Lambda'_{i^-,j^+} := \Lambda_{i-1} + \Lambda_{j+1} + \tilde{\Lambda} \in P_k^+(\Lambda).$$

Then, we draw an edge between Λ' and Λ'_{i^-,j^+}.

e.g., $P_3^+(\Lambda_0 + \Lambda_3 + \Lambda_6)$ in type $A_6^{(1)}$ consists of $\Lambda_0 + \Lambda_3 + \Lambda_6$, $\Lambda_1 + \Lambda_2 + \Lambda_6$, $\Lambda_1 + \Lambda_3 + \Lambda_5$, $\Lambda_0 + \Lambda_4 + \Lambda_5$, $\Lambda_2 + \Lambda_3 + \Lambda_4$, $2\Lambda_0 + \Lambda_2$, $\Lambda_4 + 2\Lambda_6$, $2\Lambda_5 + \Lambda_6$, $\Lambda_0 + 2\Lambda_1$, $2\Lambda_2 + \Lambda_5$, $\Lambda_1 + 2\Lambda_4$, $2\Lambda_0 + \Lambda_2$, $3\Lambda_3$.

e.g., $P_3^+(\Lambda_0+\Lambda_3+\Lambda_6)$ in type $A_6^{(1)}$ consists of $\Lambda_0+\Lambda_3+\Lambda_6$, $\Lambda_1+\Lambda_2+\Lambda_6$, $\Lambda_1+\Lambda_3+\Lambda_5$, $\Lambda_0+\Lambda_4+\Lambda_5$, $\Lambda_2+\Lambda_3+\Lambda_4$, $2\Lambda_0+\Lambda_2$, $\Lambda_4+2\Lambda_6$, $2\Lambda_5+\Lambda_6$, $\Lambda_0+2\Lambda_1$, $2\Lambda_2+\Lambda_5$, $\Lambda_1+2\Lambda_4$, $2\Lambda_0+\Lambda_2$, $3\Lambda_3$.

We then obtain



We define

$$\Delta_{i^-,j^+} := \begin{cases} (0^i, 1^{j-i+1}, 0^{\ell-j}) & \text{if } i \leq j, \\ (1^{j+1}, 0^{i-j-1}, 1^{\ell-i+1}) & \text{if } i > j. \end{cases}$$

We define

$$\Delta_{i^-,j^+} := \begin{cases} (0^i, 1^{j-i+1}, 0^{\ell-j}) & \text{if } i \leq j, \\ (1^{j+1}, 0^{i-j-1}, 1^{\ell-i+1}) & \text{if } i > j. \end{cases}$$

We draw an arrow $\Lambda' \longrightarrow \Lambda'_{i^-,j^+}$ if

$$X_{\Lambda'} + \Delta_{i^-,j^+} = X_{\Lambda'_{i^-,j^+}}$$

We define

$$\Delta_{i^-,j^+} := \begin{cases} (0^i, 1^{j-i+1}, 0^{\ell-j}) & \text{if } i \le j, \\ (1^{j+1}, 0^{i-j-1}, 1^{\ell-i+1}) & \text{if } i > j. \end{cases}$$

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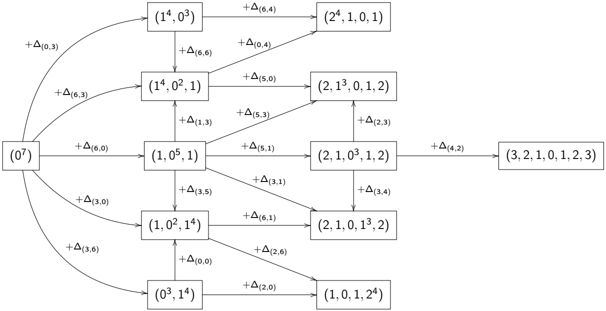


Constructions in affine type C

Recall that $P_k^+(\Lambda) = \{\Lambda' \in P^+ \mid |\Lambda| = |\Lambda'| = k, \mathrm{ev}(\Lambda) \equiv_2 \mathrm{ev}(\Lambda')\ \}$.

- Set $\Lambda' = \Lambda_i + \tilde{\Lambda} \in P_k^+(\Lambda)$. We define

$$\Lambda'_{i^+} := \Lambda_{i+2} + \tilde{\Lambda} \qquad \Lambda'_{i^-} := \Lambda_{i-2} + \tilde{\Lambda}.$$

- Set $\Lambda' = \Lambda_i + \Lambda_j + \tilde{\Lambda} \in P_k^+(\Lambda)$. We define

$$\Lambda'_{i^+,j^+} := \Lambda_{i+1} + \Lambda_{j+1} + \tilde{\Lambda} \qquad \Lambda'_{i^-,j^-} := \Lambda_{i-1} + \Lambda_{j-1} + \tilde{\Lambda}.$$

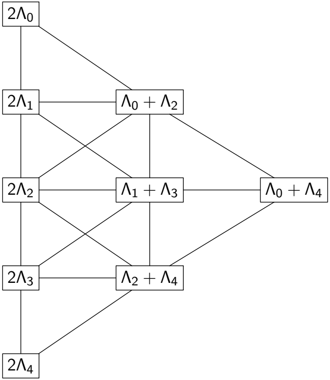
- Set $\Lambda' = \Lambda_i + \Lambda_j + \tilde{\Lambda} \in P_k^+(\Lambda)$ with $i \neq 0$, $j \neq \ell$, $i - 1 \neq j$. We define

$$\Lambda'_{i^-,j^+} := \Lambda_{i-1} + \Lambda_{j+1} + \tilde{\Lambda}$$

Then, we draw an edge between Λ' and $\Lambda'_{i^\pm}$, $\Lambda'_{i^\pm,j^\pm}$, Λ'_{i^-,j^+}.

e.g., $P_2^+(2\Lambda_2)$ in type $C_4^{(1)}$ consists of $2\Lambda_0$, $2\Lambda_1$, $2\Lambda_2$, $2\Lambda_3$, $2\Lambda_4$, $\Lambda_0+\Lambda_2$, $\Lambda_1+\Lambda_3$, $\Lambda_2+\Lambda_4$, $\Lambda_0+\Lambda_4$.

We then obtain



We define

- $\Delta_{i^+} := (1, 2^i, 1, 0^{\ell-i-1}), \quad \Delta_{i^-} := (0^{i-1}, 1, 2^{\ell-i}, 1).$
- $\Delta_{i^+,j^+} := (1, 2^i, 1^{j-i}, 0^{\ell-j}), \quad \Delta_{i^-,j^-} := (0^i, 1^{j-i}, 2^{\ell-j}, 1).$
- $\Delta_{i^-,j^+} := \begin{cases} (0^i, 1^{j-i+1}, 0^{\ell-j}) & \text{if } i \leq j, \\ (1, 2^j, 1^{i-j-1}, 2^{\ell-i}, 1) & \text{if } i \geq j+2. \end{cases}$

Set Δ and Λ'' for $\Lambda'_{i^\pm}$, $\Lambda'_{i^\pm,j^\pm}$, Λ'_{i^-,j^+}, respectively.

We define

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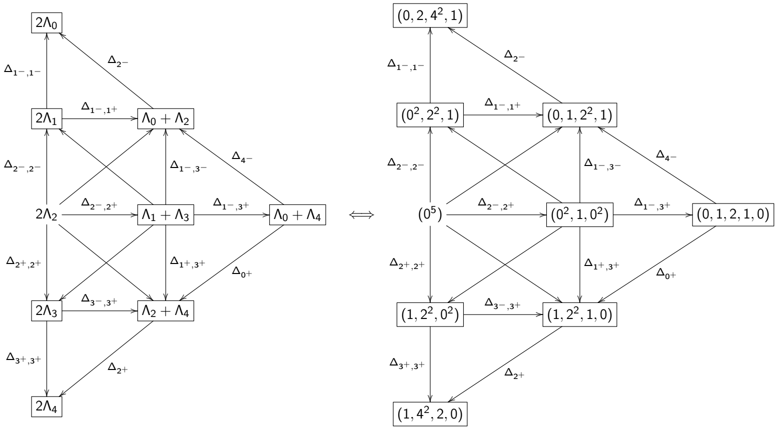
- $\Delta_{i^-,j^+} := \begin{cases} (0^i, 1^{j-i+1}, 0^{\ell-j}) & \text{if } i \le j, \\ (1, 2^j, 1^{i-j-1}, 2^{\ell-i}, 1) & \text{if } i \ge j+2. \end{cases}$

Set Δ and Λ'' for $\Lambda'_{i^\pm}$, $\Lambda'_{i^\pm,j^\pm}$, Λ'_{i^-,j^+}, respectively.

We draw an arrow $\Lambda' \longrightarrow \Lambda''$ if

$$X_{\Lambda'} + \Delta = X_{\Lambda''}.$$

e.g., the quiver for $P_2^+(2\Lambda_2)$ in type $C_4^{(1)}$ is displayed as



Rule to draw arrows

Let Δ_{fin}^{+} be the set of positive roots of the root system of type X.

- If $X = A_\ell$, $\Delta_{\mathrm{fin}}^{+} = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq \ell + 1\}$.
- If $X = B_\ell$, $\Delta_{\mathrm{fin}}^{+} = \{\epsilon_i \mid 1 \leq i \leq \ell\} \sqcup \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq \ell\}$.
- If $X = C_\ell$, $\Delta_{\mathrm{fin}}^{+} = \{2\epsilon_i \mid 1 \leq i \leq \ell\} \sqcup \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq \ell\}$.
- If $X = D_\ell$, $\Delta_{\mathrm{fin}}^{+} = \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq \ell\}$.

Rule to draw arrows

Let Δ^+_{fin} be the set of positive roots of the root system of type X.

- If $X = A_\ell$, $\Delta^+_{\mathrm{fin}} = \{\epsilon_i - \epsilon_j \mid 1 \le i < j \le \ell + 1\}$.
- If $X = B_\ell$, $\Delta^+_{\mathrm{fin}} = \{\epsilon_i \mid 1 \le i \le \ell\} \sqcup \{\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le \ell\}$.
- If $X = C_\ell$, $\Delta^+_{\mathrm{fin}} = \{2\epsilon_i \mid 1 \le i \le \ell\} \sqcup \{\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le \ell\}$.
- If $X = D_\ell$, $\Delta^+_{\mathrm{fin}} = \{\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le \ell\}$.

Then, the set $\Delta^+_{\mathrm{fin}} \sqcup (\delta - \Delta^+_{\mathrm{fin}})$ gives all arrows $\Lambda' \longrightarrow \Lambda''$.

Arrows in affine type A

Recall that $\delta = \alpha_0 + \alpha_1 + \cdots + \alpha_\ell = (1, 1, \ldots, 1)$. Then,

$$\Delta^+_{\mathrm{fin}} \sqcup (\delta - \Delta^+_{\mathrm{fin}}) = \{\epsilon_i - \epsilon_j, \delta - (\epsilon_i - \epsilon_j) \mid 1 \le i < j \le \ell + 1\}.$$

We have $\Delta_{i^-, j^+} =$

$$\begin{cases} (0^i, 1^{j-i+1}, 0^{\ell-j}) = \epsilon_i - \epsilon_{j+1} & \text{if } 0 < i \le j \le \ell, \\ (1^{j+1}, 0^{\ell-j}) = \delta - (\epsilon_{j+1} - \epsilon_{\ell+1}) & \text{if } 0 = i \le j \le \ell - 1, \\ (1^{j+1}, 0^{i-j-1}, 1^{\ell-i+1}) = \delta - (\epsilon_{j+1} - \epsilon_i) & \text{if } 0 \le j < i \le \ell. \end{cases}$$

Arrows in affine type C

Recall that $\delta = \alpha_0 + 2\alpha_1 + \cdots + 2\alpha_{\ell-1} + \alpha_\ell = (1, 2, \ldots, 2, 1)$.

- $\Delta_{i^+} = (1, 2^i, 1, 0^{\ell-i-1}) = \delta - (\epsilon_{i+1} + \epsilon_{i+2})$.

$$\Rightarrow \{\delta - (\epsilon_i + \epsilon_{i+1}) \mid 1 \leq i \leq \ell - 1\}.$$

- $\Delta_{i^-} = (0^{i-1}, 1, 2^{\ell-i}, 1) = \epsilon_{i-1} + \epsilon_i$.

$$\Rightarrow \{\epsilon_i + \epsilon_{i+1} \mid 1 \leq i \leq \ell - 1\}.$$

- $\Delta_{i^+,j^+} = (1, 2^i, 1^{j-i}, 0^{\ell-j})$ with $i + 1 \neq j$.

$$\Rightarrow \{\delta - (\epsilon_i + \epsilon_j) \mid 1 \leq i \leq j \leq \ell - 1, i + 1 \neq j\}.$$

- $\Delta_{i^-,j^-} = (0^i, 1^{j-i}, 2^{\ell-j}, 1)$ with $i + 1 \neq j$.

$$\Rightarrow \{\epsilon_i + \epsilon_j \mid 1 \leq i \leq j \leq \ell - 1, i + 1 \neq j\}.$$

- Δ_{i^-,j^+} with $i \neq 0, j \neq \ell, i - 1 \neq j$.

$$\Rightarrow \{\epsilon_i - \epsilon_j, \delta - (\epsilon_i - \epsilon_j) \mid 1 \leq i < j \leq \ell - 1\}.$$

Key Lemmas

Lemma 1

The quiver $\vec{C}(\Lambda)$ of $P_k^+(\Lambda)$ is a finite connected quiver.

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The quiver $\vec{C}(\Lambda)$ of $P_k^+(\Lambda)$ is a finite connected quiver.

Lemma 2

Suppose $\Lambda = \bar{\Lambda} + \tilde{\Lambda}$. There is a directed path

$$\Lambda^{(1)} \longrightarrow \Lambda^{(2)} \longrightarrow \cdots \longrightarrow \Lambda^{(m)} \in \vec{C}(\bar{\Lambda})$$

if and only if there is a directed path

$$\Lambda^{(1)} + \tilde{\Lambda} \longrightarrow \Lambda^{(2)} + \tilde{\Lambda} \longrightarrow \cdots \longrightarrow \Lambda^{(m)} + \tilde{\Lambda} \in \vec{C}(\Lambda).$$

Lemma 3

Suppose that there is an arrow $\Lambda' \longrightarrow \Lambda''$ in $\vec{C}(\Lambda)$. If $R^\Lambda(\beta_{\Lambda'})$ is representation-infinite (resp. wild), then so is $R^\Lambda(\beta_{\Lambda''})$.

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Suppose that there is an arrow $\Lambda' \longrightarrow \Lambda''$ in $\vec{C}(\Lambda)$. If $R^{\Lambda}(\beta_{\Lambda'})$ is representation-infinite (resp. wild), then so is $R^{\Lambda}(\beta_{\Lambda''})$.

Lemma 4

Write $\Lambda = \bar{\Lambda} + \tilde{\Lambda}$. If $R^{\bar{\Lambda}}(\beta)$ is representation-infinite (resp. wild), then $R^{\Lambda}(\beta)$ is representation-infinite (resp. wild).

Rep-finite and tame sets in affine type A

Set $i_0 := i_h$, $i_{h+1} := i_1$ and write

$$\Lambda = m_{i_1}\Lambda_{i_1} + \cdots + m_{i_j}\Lambda_{i_j} + m_{i_{j+1}}\Lambda_{i_{j+1}} + \cdots + m_{i_h}\Lambda_{i_h}$$

Rep-finite and tame sets in affine type A

Set $i_0 := i_h$, $i_{h+1} := i_1$ and write

$$\Lambda = m_{i_1}\Lambda_{i_1} + \cdots + m_{i_j}\Lambda_{i_j} + m_{i_{j+1}}\Lambda_{i_{j+1}} + \cdots + m_{i_h}\Lambda_{i_h}$$

For any $1 \le j \le h$, we define

$$F(\Lambda)_0 := \left\{\Lambda_{i_j^-, i_j^+} \mid m_{i_j} = 2\right\}$$

$$F(\Lambda)_1 := \left\{\Lambda_{i_j^-, i_{j+1}^+} \mid m_{i_j} = 1, m_{i_{j+1}} = 1\right\}$$

$$T(\Lambda)_1 := \left\{\Lambda_{i_j^-, i_{j+1}^+} \mid m_{i_j} = 1, m_{i_{j+1}} > 1 \text{ or } m_{i_j} > 1, m_{i_{j+1}} = 1\right\}$$

$$T(\Lambda)_2 := \left\{(\Lambda_{i_j^-, i_j^+})_{(i_j-1)^-, (i_j+1)^+} \mid m_{i_j} = 2, i_{j-1} \not\equiv_{\ell+1} i_j - 1, i_{j+1} \not\equiv_{\ell+1} i_j + 1\right\} \text{ if char } K \neq 2$$

$$T(\Lambda)_3 := \left\{(\Lambda_{i_j^-, i_j^+})_{i_j^-, (i_j+1)^+ \text{ or } (i_j-1)^-, i_j^+} \mid m_{i_j} = 3, i_{j+1} \not\equiv_{\ell+1} i_j + 1 \text{ or } i_{j-1} \not\equiv_{\ell+1} i_j - 1\right\}$$
$$\text{if char } K \neq 3$$

$$T(\Lambda)_4 := \left\{(\Lambda_{i_j^-, i_j^+})_{i_j^-, i_j^+} \mid m_{i_j} = 4\right\} \text{ if char } K \neq 2$$

$$T(\Lambda)_5 := \left\{(\Lambda_{i_j^-, i_j^+})_{i_p^-, i_p^+} \mid m_{i_j} = m_{i_p} = 2, i_p \not\equiv_{\ell+1} i_j \pm 1, j \neq p\right\}$$

Set

$$\mathcal{F}(\Lambda) := \{\beta_{\Lambda'} \mid \Lambda' \in \{\Lambda\} \cup F(\Lambda)_0 \cup F(\Lambda)_1\},$$

$$\mathcal{T}(\Lambda) := \{\beta_{\Lambda'} \mid \Lambda' \in \cup_{1 \leq j \leq 5} T(\Lambda)_j\}.$$

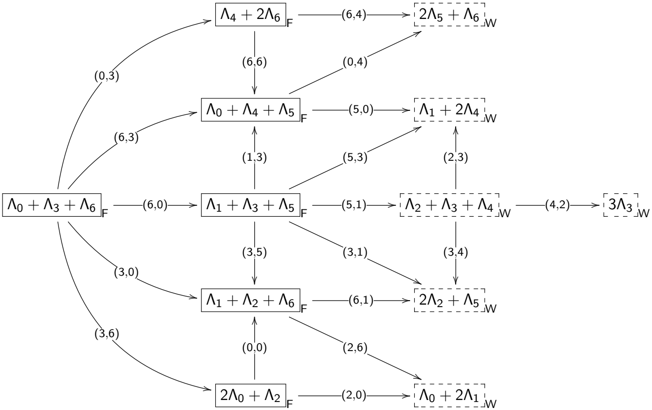
Theorem (Ariki-Song-W., 2023)

Suppose $|\Lambda| \geq 3$. Then, $R^{\Lambda}(\beta)$ is representation-finite if $\beta \in \mathcal{F}(\Lambda)$, tame if one of the following holds:

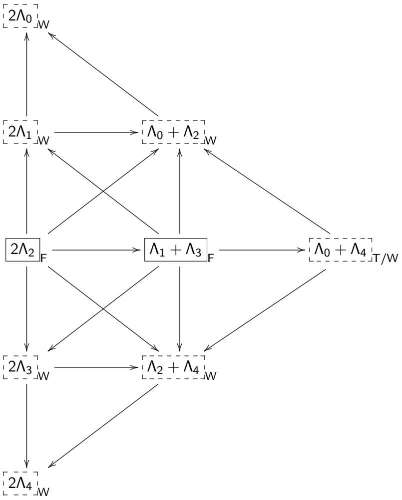
- $\beta = \delta$, $\Lambda = k\Lambda_i$, $\ell = 1$ with $t \neq \pm 2$,
- $\beta = \delta$, $\Lambda = k\Lambda_i$, $\ell \geq 2$ with $t \neq (-1)^{\ell+1}$,
- $\beta \in \mathcal{T}(\Lambda)$.

Otherwise, it is wild.

e.g., rep-type of $\vec{C}(\Lambda_0+\Lambda_3+\Lambda_6)$ in type $A_6^{(1)}$ is displayed as



e.g., rep-type of $\vec{C}(2\Lambda_2)$ in type $C_4^{(1)}$ is displayed as



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Thank you! Any questions?

Tools
- Symmetric groups and Hecke algebras;
- Bound quiver algebras;
- Representation type: rep-finite, tame, wild;
- Brauer tree/graph algebras.

Objects
- Lie theoretic data;
- Quiver Hecke algebras;
- Cyclotomic KLR algebras;
- $\max^{+}(\Lambda)$ and $P_k^{+}(\Lambda)$;
- Rep-finite and tame sets.