On τ-tilting finiteness of Schur algebras

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Introduction

As a generalization of the classical tilting modules, Adachi, Iyama and Reiten introduced the support τ-tilting modules, which have many nice properties. For example,

- these modules are bijectively corresponding to two-term silting complexes, left finite semibricks and so on.
- there is a partial order on the set of isomorphism classes of basic support τ-tilting modules such that its Hasse quiver realizes the left mutation. (See Proposition 1.4 in this talk.)

An algebra A over an algebraically closed field K is called τ-tilting finite if there are only finitely many pairwise non-isomorphic basic support τ-tilting A-modules. Otherwise, A is called τ-tilting infinite.

Background

(1) The τ-tilting finiteness for several classes of algebras is known, such as

- preprojective algebras of Dynkin type (Mizuno, 2014);
- algebras with radical square zero (Adachi, 2016);
- Brauer graph algebras (Adachi-Aihara-Chan, 2018);
- biserial algebras (Mousavand, 2019);
- minimal wild two-point algebras (W, 2019).

(2) Representation-finite algebras are τ-tilting finite and the converse is not true in general. For example, let

$$\Lambda_n := K(\circ \xrightarrow{a} \circ \circlearrowleft b)/ < b^n, ab^2 >, \ n \geqslant 2,$$

then Λ_n is τ-tilting finite (by direct calculation). But Λ_n is representation-finite if $n \leqslant 5$; tame if $n = 6$; wild if $n \geqslant 7$.

However, the converse is true in some special cases, including

- cycle-finite algebras (Malicki-Skowroński, 2016);
- gentle algebras (Plamondon, 2018);
- tilted and cluster-tilted algebras (Zito, 2019);
- simply connected algebras (W, 2019);
- quasi-tilted algebras, locally hereditary algebras, etc., (Aihara-Honma-Miyamoto-W, 2020).

The subject of this talk

Let n, r be positive integers and $\mathbb{F}$ an algebraically closed field of characteristic p. We take an n-dimensional vector space V over $\mathbb{F}$ with a basis $\{v_1, v_2, \ldots, v_n\}$. Then, the r-fold tensor product $V^{\otimes r} := V \otimes_{\mathbb{F}} V \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} V$ has a $\mathbb{F}$-basis given by

$$\{v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r} \mid 1 \leqslant i_j \leqslant n \text{ for all } 1 \leqslant j \leqslant r\}.$$

Let G_r be the symmetric group on r symbols. Then,

$$(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r}) \cdot \sigma = v_{i_{\sigma(1)}} \otimes v_{i_{\sigma(2)}} \otimes \cdots \otimes v_{i_{\sigma(r)}}.$$

for any $\sigma \in G_r$. We define the Schur algebra $S(n, r)$ by the endomorphism ring $\mathrm{End}_{\mathbb{F}G_r}(V^{\otimes r})$.

In this talk, we will discuss the τ-tilting finiteness of $S(n,r)$, and see when the condition

$$\tau\text{-titling finite} \Leftrightarrow \text{representation-finite}$$

is true on the class of Schur algebras.

τ-tilting theory

Auslander-Reiten translation

Let A be a finite-dimensional basic algebra over an algebraically closed field K. For an A-module M with a minimal projective presentation

$$P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0,$$

the transpose of M is given by

$$\mathrm{Tr}\ M := \mathsf{coker}\ \mathsf{Hom}_A(d_1, A).$$

Then, the Auslander-Reiten translation is defined by

$$\tau(M) := D\mathrm{Tr}\ M,$$

where $D = \mathsf{Hom}_K(-, K)$ is the standard K-duality.

We denote by $|M|$ the number of isomorphism classes of indecomposable direct summands of M.

Definition 1.1 (Adachi-Iyama-Reiten, 2014)

Let M be a right A-module and P a projective A-module.

1. M is called τ-rigid if $\mathrm{Hom}_A(M, \tau M) = 0$.
2. M is called τ-tilting if M is τ-rigid and $|M| = |A|$.
3. M is called support τ-tilting if there exists an idempotent e of A such that M is a τ-tilting $(A/\langle e\rangle)$-module.

We denote by τ-rigid A (resp., sτ-tilt A) the set of isomorphism classes of indecomposable τ-rigid (resp., basic support τ-tilting) A-modules.

Example

Let $A := K(\ 1 \underset{b}{\overset{a}{\rightleftarrows}} 2\ )/ < ab, ba >$. We denote by S_i the simple A-modules and P_i the indecomposable projective A-modules. Then, we have

$$\tau(S_1) = S_2, \tau(S_2) = S_1, \tau(P_1) = 0, \tau(P_2) = 0.$$

Thus,

- τ-rigid $A = \{P_1, P_2, S_1, S_2, 0\}$;
- sτ-tilt $A = \{P_1 \oplus P_2, P_1 \oplus S_1, S_2 \oplus P_2, S_1, S_2, 0\}$;
- $P_1 \oplus P_2, P_1 \oplus S_1$ and $S_2 \oplus P_2$ are τ-tilting modules.

We denote by add(M) (resp., Fac(M)) the full subcategory whose objects are direct summands (resp., factor modules) of finite direct sums of copies of M.

A morphism $\pi : M \to N'$ is called a minimal left add(N)-approximation of M if $N' \in \mathrm{add}(N)$ and it satisfies:

(1) any $h : N' \to N'$ satisfying $h \circ \pi = \pi$ is an automorphism.

$$\begin{array}{ccc} M & \xrightarrow{\ \pi\ } & N' \\ & \searrow^{\pi} & \downarrow h \simeq \mathrm{id} \\ & & N' \end{array}$$

(2) for any $N'' \in \mathrm{add}(N)$ and $g : M \to N''$, there exists $f : N' \to N''$ such that $f \circ \pi = g$.

$$\begin{array}{ccc} M & \xrightarrow{\ \pi\ } & N' \\ & \searrow^{\forall g} & \downarrow \exists f \\ & & N'' \end{array}$$

Mutation

Definition 1.2 (Adachi-Iyama-Reiten, 2014)

Let $T = M \oplus N$ be a basic support τ-tilting A-module with an indecomposable direct summand M satisfying $M \notin \mathsf{Fac}(N)$. We take a minimal left $\mathsf{add}(N)$-approximation π with an exact sequence

$$M \xrightarrow{\pi} N' \longrightarrow \operatorname{coker} \pi \longrightarrow 0.$$

Then, we call $\mu_M^-(T) := \operatorname{coker} \pi \oplus N$ the left mutation of T with respect to M, which is again a basic support τ-tilting A-module.

Example

Let $A = K(1 \underset{b}{\overset{a}{\rightleftarrows}} 2)/ < ab, ba >$, then $P_1 \oplus P_2$ is a τ-tilting module. We consider the left mutation with respect to P_2,

$$P_2 \xrightarrow{\pi} P_1 \longrightarrow \text{coker } \pi \longrightarrow 0,$$

where $\pi : \begin{smallmatrix} e_2 \\ b \end{smallmatrix} \xrightarrow{a\cdot} \begin{smallmatrix} e_1 \\ a \end{smallmatrix}$ is a minimal left add(P_1)-approximation. Then,

$$\text{coker } \pi = S_1 \text{ and } \mu^-_{P_2}(A) = P_1 \oplus S_1.$$

In fact, we have the following mutation quiver of sτ-tilt A.

$$\begin{array}{ccccc}
P_1 \oplus P_2 & \longrightarrow & P_1 \oplus S_1 & \longrightarrow & S_1 \\
\downarrow & & & & \downarrow \\
S_2 \oplus P_2 & \longrightarrow & S_2 & \longrightarrow & 0
\end{array}$$

Poset structure on sτ-tilt A

Definition 1.3 (Adachi-Iyama-Reiten, 2014)

For $M, N \in$ sτ-tilt A, we say $M \geqslant N$ if $\mathrm{Fac}(N) \subseteq \mathrm{Fac}(M)$.

Example

Let A be the previous algebra. The Hasse quiver of sτ-tilt A is

$$\begin{array}{ccccc} P_1 \oplus P_2 & \xrightarrow{>} & P_1 \oplus S_1 & \xrightarrow{>} & S_1 \\ \downarrow{\scriptstyle >} & & & & \downarrow{\scriptstyle >} \\ S_2 \oplus P_2 & \xrightarrow{>} & S_2 & \xrightarrow{>} & 0 \end{array}$$

Proposition 1.4 (Adachi-Iyama-Reiten, 2014)

The mutation quiver $\mathcal{Q}(\mathsf{s}\tau\text{-tilt } A)$ and the Hasse quiver $\mathcal{H}(\mathsf{s}\tau\text{-tilt } A)$ coincide.

Proposition 1.5 (Adachi-Iyama-Reiten, 2014)

If the Hasse quiver $\mathcal{H}(\mathsf{s}\tau\text{-tilt } A)$ contains a finite connected component, then it exhausts all pairwise non-isomorphic basic support τ-tilting A-modules.

Reduction theorems

Proposition 1.6 (Demonet-Iyama-Jasso, 2017)

If A is τ-tilting finite, then

(1) the quotient A/I is τ-tilting finite for any two-sided ideal I of A.

(2) the idempotent truncation eAe is τ-tilting finite for any idempotent e of A.

Proposition 1.7 (Eisele-Janssens-Raedschelders, 2018)

Let I be a two-sided ideal generated by central elements which are contained in the Jacobson radical of A. Then, there exists a poset isomorphism between sτ-tilt A and sτ-tilt (A/I).

Schur algebras

Permutation modules

Let $\mathbb{F}$ be an algebraically closed field of characteristic p.

- r: a positive number.
- $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$: a partition of r with at most n parts.
- $\Omega(n, r)$: the set of all partitions of r with at most n parts.
- G_r: the symmetric group on r symbols.
- $\mathbb{F}G_r$: the group algebra of G_r over $\mathbb{F}$.

We may define the Young subgroup G_λ of G_r associated with a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ of r by

$$G_\lambda := G_{\lambda_1} \times G_{\lambda_2} \times \ldots \times G_{\lambda_n}.$$

Then, the permutation $\mathbb{F}G_r$-module M^λ is $1_{G_\lambda} \uparrow^{G_r}$, where 1_{G_λ} denotes the trivial module for G_λ and $\uparrow$ denotes the induction.

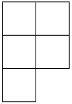
Let $S(n,r) = \mathrm{End}_{\mathbb{F}G_r}(V^{\otimes r})$ be the Schur algebra. We have the following well-known algebra isomorphism

$$S(n,r) \simeq \mathrm{End}_{\mathbb{F}G_r}\left(\bigoplus_{\lambda\in\Omega(n,r)} n_\lambda M^\lambda\right),$$

where $1 \leqslant n_\lambda \in \mathbb{N}$ is the number of compositions of r with at most n parts which are rearrangement of λ.

Specht modules and the decomposition matrix

Let p be a prime and $[\lambda]$ the Young diagram of partition λ. We call λ a *p*-regular partition if no p rows of $[\lambda]$ have the same length. Otherwise, λ is called p-singular.

A 3-regular partition: $[(2,2,1)] =$  .

Let S^λ be the Specht module of $\mathbb{F}G_r$ corresponding to λ.

- If char $\mathbb{F} = 0$, then $\left\{ S^\lambda \mid \lambda: \text{ partition of } r \right\}$ is a complete set of pairwise non-isomorphic simple $\mathbb{F}G_r$-modules.
- If char $\mathbb{F} = p$, then $\left\{ D^\lambda \mid \lambda: \ p\text{-regular partition of } r \right\}$ is a complete set of pairwise non-isomorphic simple $\mathbb{F}G_r$-modules, where D^λ is the unique simple top $S^\lambda/(\text{rad } S^\lambda)$ of S^λ with λ being p-regular.

In the case of a p-singular partition μ, all of the composition factors of S^μ are D^λ such that λ is a p-regular partition which dominates μ. Therefore, the decomposition matrix of $\mathbb{F}G_r$ has the following form (see [James, 1978]),

$$\begin{array}{r} S^\lambda, \lambda\ p\text{-regular} \left\{ \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \right. \\ \\ S^\lambda, \lambda\ p\text{-singular} \left\{ \begin{array}{c} \\ \\ \end{array} \right. \end{array} \overbrace{\begin{pmatrix} 1 & & & & \\ * & 1 & & O & \\ * & * & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ * & * & * & \dots & 1 \\ - & - & - & - & - \\ * & * & * & \dots & * \\ * & * & * & \dots & * \end{pmatrix}}^{D^\mu,\ \mu\ p\text{-regular}}.$$

Young modules

Let $\mathbb{F}$ be an algebraically closed field and p a prime.

- χ^λ: the ordinary character corresponding to Specht module S^λ over char $\mathbb{F} = 0$.
- ch M^λ: the associated ordinary character of permutation module M^λ over char $\mathbb{F} = p$.

Then, χ^λ is a constituent of ch M^λ and $\chi^\mu(\mu \neq \lambda)$ is a constituent of ch M^λ if and only if $\mu \rhd \lambda$.

Let $M^\lambda := \oplus_{i=1}^n Y_i$ with $n \in \mathbb{N}$. The unique direct summand Y_i which the ordinary character χ^λ occurs in ch Y_i, is called the Young module corresponding to λ and we denote it by Y^λ.

Let Y^λ be the Young module corresponding to λ.

- $Y^\lambda = (Y^\lambda)^*$ with $(-)^* := \mathrm{Hom}(-, \mathbb{F})$.
- Y^λ has a Specht filtration: $Y^\lambda = Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_k = 0$ for some $k \in \mathbb{N}$ with each Z_i/Z_{i+1} isomorphic to a Specht module S^μ with $\mu \trianglerighteq \lambda$.
- If λ is a partition with at most n parts, then each composition factor D^μ of Y^λ is also corresponding to the partition μ with at most n parts.

Proposition 2.1 (Martin, 1993)

The set $\{ Y^\lambda \mid \lambda \in \Omega(n, r) \}$ is a complete set of pairwise non-isomorphic Young modules which occurs as indecomposable direct summands of permutation modules M^λ with $\lambda \in \Omega(n, r)$.

The basic algebra of $S(n, r)$

Let B be a block of $\mathbb{F}G_r$ labeled by a p-core ω, it is well-known that a partition λ belongs to B if and only if λ has the same p-core ω. Then, we define

$$S_B := \mathrm{End}_{\mathbb{F}G_r}\left(\bigoplus_{\lambda\in B\cap\Omega(n,r)} Y^{\lambda}\right)$$

and the basic algebra of $S(n, r)$ is $\bigoplus S_B$, where the sum is taken over all blocks of $\mathbb{F}G_r$. Moreover, S_B is a direct sum of blocks of the basic algebra of $S(n, r)$.

Reduction theorems on $S(n,r)$

Lemma 2.2 (W, 2020)

If $S(n,r)$ is τ-tilting infinite, then so is $S(N,r)$, for any $N > n$.

Sketch of the proof: We show that $S(n,r)$ is an idempotent truncation of $S(N,r)$.

Lemma 2.3 (W, 2020)

If $S(n,r)$ is τ-tilting infinite, then so is $S(n,n+r)$.

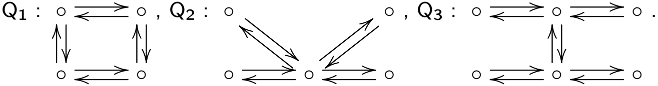
Sketch of the proof: It is shown by [Erdmann, 1993] that $S(n,r)$ is a quotient of $S(n,n+r)$.

Strategy on τ-tilting infinite Schur algebras

Let $A := \mathbb{F}Q/I$ be an algebra. We call Q a τ-tilting infinite quiver if $A/\mathrm{rad}^2 A$ is τ-tilting infinite. For example, the Kronecker quiver $Q:\ \circ \rightrightarrows \circ$ is a τ-tilting infinite quiver.

Lemma 2.4 (W, 2020)

The following Q_1, Q_2 and Q_3 are τ-tilting infinite quivers.



Sketch of the proof: We remark that representation-infinite path algebras are τ-tilting infinite.

Useful information on $S(2,r)$

A partition (λ_1, λ_2) of r is uniquely determined by $s = \lambda_1 - \lambda_2$.

- v^s: the vertex in the quiver of the basic algebra of $S(2,r)$ corresponding to the Young module $Y^{(\lambda_1,\lambda_2)}$.
- $n(v^s, v^t)$: the number of arrows from v^s to v^t.

Theorem 2.5 (Erdmann-Henke, 2002)

We have $n(v^s, v^t) = n(v^t, v^s)$. Let $s = s_0 + ps'$ and $t = t_0 + pt'$ with $0 \leqslant s_0, t_0 \leqslant p-1$ and $s', t' \geqslant 0$. Suppose $s > t$.

(1) If $p = 2$, then

$$n(v^s, v^t) = \begin{cases} n(v^{s'}, v^{t'}) & \text{if } s_0 = t_0 = 1 \text{ or } s_0 = t_0 = 0 \text{ and } s' \equiv t' \mod 2, \\ 1 & \text{if } s_0 = t_0 = 0, t' + 1 = s' \not\equiv 0 \mod 2, \\ 0 & \text{otherwise.} \end{cases}$$

(2) If $p > 2$, then

$$n(v^s, v^t) = \begin{cases} n(v^{s'}, v^{t'}) & \text{if } s_0 = t_0, \\ 1 & \text{if } s_0 + t_0 = p - 2, t' + 1 = s' \not\equiv 0 \mod p, \\ 0 & \text{otherwise.} \end{cases}$$

Example 1

Let $p = 2$. We consider the quiver of the basic algebra of $S(2, 10)$. By Theorem 2.5, we have

$$\begin{array}{ccccccc} & & & & (6,4) & \rightleftarrows & (10) \\ & & & & \uparrow\downarrow & & \uparrow\downarrow \\ (8,2) & \rightleftarrows & (7,3) & \rightleftarrows & (5^2) & \rightleftarrows & (9,1) \end{array} .$$

It is obvious that $S(2, 10)$ is τ-tilting infinite.

Example 2

Let $p = 2$. The quiver of the basic algebra of $S(2, 11)$ is

$$(10,1) \rightleftarrows (6,5) \rightleftarrows (8,3) \qquad (11) \rightleftarrows (7,4) \qquad (9,2) .$$

We have to find the explicit structures of Young modules.

Let p be a prime. For any non-negative integer s, there is a p-adic decomposition $s = \sum_{k\geqslant 0} s_k p^k$. Now, let s, t be two non-negative integers, we define a function

$$f(s,t) = \prod_{k\in\{0\}\cup\mathbb{N}} \binom{p-1-s_k}{p-1-t_k}.$$

Moreover, we have

$$g(s,t) := \begin{cases} 1 & \text{if } f(2t, s+t) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$h(s,t) := \begin{cases} 1 & \text{if } f(2t+1, s+t+1) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.6 (Henke, 1999)

Let $(r-k,k)$ be a partition with a non-negative integer k.

(1) If r is even, then

$$\text{ch } Y^{(r-k,k)} = \sum_{i=0}^{\frac{r}{2}} g(\tfrac{r}{2}-i, \tfrac{r}{2}-k)\chi^{(r-i,i)}.$$

(2) If r is odd, then

$$\text{ch } Y^{(r-k,k)} = \sum_{i=0}^{[\frac{r}{2}]} h([\tfrac{r}{2}]-i, [\tfrac{r}{2}]-k)\chi^{(r-i,i)},$$

where $[\frac{r}{2}]$ is the greatest integer less than $\frac{r}{2}$.

Suppose $p = 2$. Let B_1 be the principal block of $\mathbb{F}G_{11}$ and B_2 the block of $\mathbb{F}G_{11}$ labeled by 2-core $(2,1)$, the parts of the decomposition matrix $[S^\lambda : D^\mu]$ (see [James, 1978]) for the partitions in B_1 and B_2 with at most two parts are

$$B_1 : \begin{matrix}(11)\\(9,2)\\(7,4)\end{matrix}\begin{pmatrix}1 & & \\ 0 & 1 & \\ 1 & 0 & 1\end{pmatrix},\ B_2 : \begin{matrix}(10,1)\\(8,3)\\(6,5)\end{matrix}\begin{pmatrix}1 & & \\ 1 & 1 & \\ 0 & 1 & 1\end{pmatrix}.$$

We first determine S_{B_2}. By Theorem 2.6, we have

$$\begin{aligned}\text{ch } Y^{(10,1)} &= \chi^{(10,1)},\\ \text{ch } Y^{(8,3)} &= \chi^{(10,1)} + \chi^{(8,3)},\\ \text{ch } Y^{(6,5)} &= \chi^{(10,1)} + \chi^{(8,3)} + \chi^{(6,5)}.\end{aligned}$$

Recall that $\dim \text{Hom}_{\mathbb{F}G_{11}}(Y^\lambda, Y^\mu)$ is equal to the inner product $(\text{ch } Y^\lambda, \text{ch } Y^\mu)$. Then, we have

- $Y^{(10,1)} = D^{(10,1)}$, $Y^{(8,3)} = \begin{array}{c} D^{(10,1)} \\ D^{(8,3)} \\ D^{(10,1)} \end{array}$,
- $Y^{(6,5)} = \begin{array}{ccccc} & D^{(8,3)} & & & D^{(10,1)} \\ \diagup & & \diagdown & & \diagup \\ D^{(10,1)} & & D^{(6,5)} & & \\ & & & \diagdown & \\ & & & & D^{(8,3)} \end{array}$.

Hence, $S_{B_2} = \mathrm{End}_{\mathbb{F}G_{11}}(Y^{(10,1)} \oplus Y^{(8,3)} \oplus Y^{(6,5)})$ is isomorphic to $\mathbb{F}Q/I$ with

$$Q: \ (10,1) \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} (6,5) \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} (8,3) \ \text{ and}$$

$$I: \langle \alpha_1\beta_1, \beta_2\alpha_2, \alpha_1\alpha_2\beta_2, \alpha_2\beta_2\beta_1 \rangle.$$

Similarly, $S_{B_1} = \mathbb{F}\left((11) \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} (7,4) \qquad (9,2) \right) / \langle \alpha_1\beta_1 \rangle$.

We point out the following facts.

- S_B is uniquely determined by the decomposition matrix $[S^\lambda : D^\mu]$ and the character ch Y^λ of Young modules;
- We may compute the basic algebra of $S(2, r)$ by hand.
- We may compute the basic algebra of $S(n, r)$ with $n \geqslant 3$ by a computer. For example, we refer to Carlson and Matthews's program:

http://alpha.math.uga.edu/ jfc/schur.html.

Tame Schur algebras

The complete classification of representation type

The representation type of Schur algebras is completely determined by [Erdmann, 1993], [Xi, 1993], [Doty-Nakano, 1998] and [Doty-Erdmann-Martin-Nakano, 1999].

Proposition 2.7

Let p be the characteristic of $\mathbb{F}$. Then, $S(n, r)$ is

- semi-simple if and only if $p = 0$ or $p > r$ or $p = 2, n = 2, r = 3$;
- representation-finite if and only if $p = 2, n = 2, r = 5, 7$ or $p \geqslant 2, n = 2, r < p^2$ or $p \geqslant 2, n \geqslant 3, r < 2p$;
- infinite-tame if and only if $p = 2, n = 2, r = 4, 9, 11$ or $p = 3, n = 2, r = 9, 10, 11$ or $p = 3, n = 3, r = 7, 8$.

Otherwise, $S(n, r)$ is wild.

Representation-finite blocks of Schur algebras

Proposition 2.8 (Erdmann, 1993; Donkin-Reiten, 1994)

Let A be a representation-finite block of Schur algebras, then it is Morita equivalent to $\mathcal{A}_m := \mathbb{F}Q/I$ for some $m \in \mathbb{N}$, where

$$Q: \ 1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \cdots \underset{\beta_{m-2}}{\overset{\alpha_{m-2}}{\rightleftarrows}} m-1 \underset{\beta_{m-1}}{\overset{\alpha_{m-1}}{\rightleftarrows}} m \ ,$$

$$I: \langle \alpha_1\beta_1, \alpha_i\alpha_{i+1}, \beta_{i+1}\beta_i, \beta_i\alpha_i - \alpha_{i+1}\beta_{i+1} \mid 1 \leqslant i \leqslant m-2 \rangle.$$

Theorem 2.9 (W, 2020)

We have $\#\text{s}\tau\text{-tilt}\, \mathcal{A}_m = \binom{2m}{m}$.

Sketch of the proof: We remark that $\mathcal{A}_m$ is a quotient of a Brauer line algebra modulo the two-sided ideal generated by the central element $\alpha_1\beta_1$. Then, the number follows from the result by [Aoki, 2018].

Tame blocks of tame Schur algebras

Proposition 2.10 (Doty-Erdmann-Martin-Nakano, 1999)

Let A be an infinite-tame block of tame Schur algebras, then it is Morita equivalent to one of $\mathcal{D}_3$, $\mathcal{D}_4$, $\mathcal{R}_4$ and $\mathcal{H}_4$, where

- Let $\mathcal{D}_3 := \mathbb{F}Q/I$ be the special biserial algebra given by

$$Q: \ \circ \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \circ \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \circ \ \text{ and } I: \langle \alpha_1\beta_1, \beta_2\alpha_2, \alpha_1\alpha_2\beta_2, \alpha_2\beta_2\beta_1 \rangle.$$

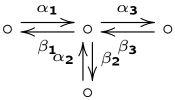
- Let $\mathcal{D}_4 := \mathbb{F}Q/I$ be the bound quiver algebra given by

$$Q: \ \begin{array}{ccccc} \circ & \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} & \circ & \underset{\alpha_3}{\overset{\beta_3}{\rightleftarrows}} & \circ \\ & & {\scriptstyle\alpha_2}\uparrow\downarrow{\scriptstyle\beta_2} & & \\ & & \circ & & \end{array} \ \text{ and } I: \left\langle \begin{array}{c} \alpha_1\beta_1, \alpha_2\beta_2, \alpha_3\beta_1, \alpha_3\beta_2, \alpha_1\beta_3, \alpha_2\beta_3, \\ \alpha_1\beta_2\alpha_2, \beta_2\alpha_2\beta_1, \beta_2\alpha_2 - \beta_3\alpha_3 \end{array} \right\rangle.$$

- Let $\mathcal{R}_4 := \mathbb{F}Q/I$ be the bound quiver algebra given by

$$Q: \ \circ \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \circ \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \circ \underset{\beta_3}{\overset{\alpha_3}{\rightleftarrows}} \circ \ \text{ and } I: \left\langle \begin{array}{c} \alpha_1\beta_1, \alpha_1\alpha_2, \beta_2\beta_1, \\ \alpha_2\beta_2 - \beta_1\alpha_1, \alpha_3\beta_3 - \beta_2\alpha_2 \end{array} \right\rangle.$$

- Let $\mathcal{H}_4 := \mathbb{F}Q/I$ be the bound quiver algebra given by

Q :


and $$I : \left\langle \begin{matrix} \alpha_1\beta_1, \alpha_1\beta_2, \alpha_2\beta_1, \alpha_2\beta_2, \alpha_1\alpha_3, \\ \beta_3\beta_1, \alpha_3\beta_3 - \beta_1\alpha_1 - \beta_2\alpha_2 \end{matrix} \right\rangle.$$

Lemma 2.11 (W, 2020)

The algebras $\mathcal{D}_3$, $\mathcal{D}_4$, $\mathcal{R}_4$ and $\mathcal{H}_4$ are τ-tilting finite. Moreover,

A	$\mathcal{D}_3$	$\mathcal{D}_4$	$\mathcal{R}_4$	$\mathcal{H}_4$
$\#$sτ-tilt A	28	114	88	96

.

Sketch of the proof: We show this lemma by direct computation with some reduction steps and software.

Theorem 2.12 (W, 2020)

If the Schur algebra $S(n, r)$ is tame, then it is τ-tilting finite.

Sketch of the proof: We have showed that $\mathcal{A}_m$, $\mathcal{D}_3$, $\mathcal{D}_4$, $\mathcal{R}_4$ and $\mathcal{H}_4$ are τ-tilting finite. It suffices to make clear that these are all the blocks of tame Schur algebras. For example, let $p = 3$, then

- the basic algebra of $S(2, 9)$ is isomorphic to $\mathcal{D}_4 \oplus \mathbb{F}$;
- the basic algebra of $S(2, 10)$ is isomorphic to $\mathcal{D}_4 \oplus \mathbb{F} \oplus \mathbb{F}$;
- the basic algebra of $S(2, 11)$ is isomorphic to $\mathcal{D}_4 \oplus \mathcal{A}_2$;
- the basic algebra of $S(3, 7)$ is isomorphic to $\mathcal{R}_4 \oplus \mathcal{A}_2 \oplus \mathcal{A}_2$;
- the basic algebra of $S(3, 8)$ is isomorphic to $\mathcal{R}_4 \oplus \mathcal{H}_4 \oplus \mathcal{A}_2$.

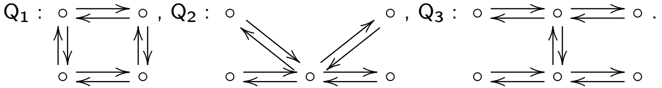
Wild Schur algebras

We consider wild Schur algebras except for the following cases.

$$(\star)\quad \begin{cases} p=2, n=2, r=8,17,19; \\ p=2, n=3, r=4; \\ p=2, n\geqslant 5, r=5; \\ p\geqslant 5, n=2, p^2\leqslant r\leqslant p^2+p-1. \end{cases}$$

A quick review:

- $S(n,r)$ τ-tilting infinite
  $\Rightarrow S(N,r)$ $(N>n)$ τ-tilting infinite
  $\Rightarrow S(n,n+r)$ τ-tilting infinite
- The following Q_1, Q_2 and Q_3 are τ-tilting infinite quivers.



The characteristic $p = 2$

Blue: τ-tilting finite **Red**: τ-tilting infinite

r	1	2	3	4	5	6	7	8	9	10	11	12
$S(2,r)$	S	F	S	T	F	W	F	W	T	W	T	W

r	13	14	15	16	17	18	19	20	21	22	23	$\cdots$
$S(2,r)$	W	W	W	W	W	W	W	W	W	W	W	$\cdots$

Sketch of the proof: We show the following.

- the basic algebra $\overline{S(2,6)}$ of $S(2,6)$ is τ-tilting finite;
- the basic algebra of $S(2,10)$ is τ-tilting infinite ($\Leftarrow$ Q$_1$);
- the basic algebra of $S(2,13)$ is $\overline{S(2,6)} \oplus \mathcal{A}_2 \oplus \mathbb{F}$;
- the basic algebra of $S(2,15)$ is $\overline{S(2,6)} \oplus \mathcal{A}_2 \oplus \mathbb{F} \oplus \mathbb{F}$;
- the basic algebra of $S(2,21)$ is τ-tilting infinite ($\Leftarrow$ Q$_1$).

The characteristic $p = 2$

n \ r	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
3	S	F	F	W	W	W	W	W	W	W	W	W	W	$\cdots$
4	S	F	F	W	W	W	W	W	W	W	W	W	W	$\cdots$
5	S	F	F	W	W	W	W	W	W	W	W	W	W	$\cdots$
6	S	F	F	W	W	W	W	W	W	W	W	W	W	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Sketch of the proof:

- the basic algebra of $S(3,6)$ is τ-tilting infinite ($\Leftarrow$ Q$_1$);
- the basic algebra of $S(3,7)$ is τ-tilting infinite ($\Leftarrow$ Q$_1$);
- the basic algebra of $S(3,8)$ is τ-tilting infinite ($\Leftarrow$ Q$_2$);
- the basic algebra of $S(4,4)$ is τ-tilting infinite ($\Leftarrow$ Q$_1$);
- the basic algebra of $S(4,5)$ is τ-tilting finite.

The characteristic $p = 3$

n \ r	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
2	S	S	F	F	F	F	F	F	T	T	T	W	W	$\cdots$
3	S	S	F	F	F	W	T	T	W	W	W	W	W	$\cdots$
4	S	S	F	F	F	W	W	W	W	W	W	W	W	$\cdots$
5	S	S	F	F	F	W	W	W	W	W	W	W	W	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Sketch of the proof:

- the basic algebras of $S(2,12)$ and $S(2,13)$ are τ-tilting infinite ($\Leftarrow Q_1$);
- the basic algebra of $S(3,6)$ is τ-tilting infinite ($\Leftarrow Q_1$);
- the basic algebras of $S(3,10)$ and $S(3,11)$ are τ-tilting infinite ($\Leftarrow Q_3$);
- the basic algebra of $S(4,7)$ is τ-tilting infinite ($\Leftarrow Q_2$);
- the basic algebra of $S(4,8)$ is τ-tilting infinite ($\Leftarrow Q_1$).

The characteristic $p \geqslant 5$

n \ r	$1 \sim p-1$	$p \sim 2p-1$	$2p \sim p^2-1$	$p^2 \sim p^2+p-1$	$p^2+p \sim \infty$
2	S	F	F	W	W
3	S	F	W	W	W
4	S	F	W	W	W
5	S	F	W	W	W
⋮	⋮	⋮	⋮	⋮	⋮

Sketch of the proof:

- the basic algebras of $S(2, p^2+p)$ and $S(2, p^2+p+1)$ are τ-tilting infinite ($\Leftarrow Q_1$);
- the basic algebras of $S(3, 2p)$, $S(3, 2p+1)$ and $S(3, 2p+2)$ are τ-tilting infinite ($\Leftarrow Q_3$).

Theorem 2.13 (W, 2020)

Let $p \geqslant 5$ and except for the cases in $(\star)$, the Schur algebra $S(n, r)$ satisfies the condition:

$$\tau\text{-titling finite} \Leftrightarrow \text{representation-finite}$$

Theorem 2.14 (W, 2020)

Let $p = 3$, the Schur algebra $S(n, r)$ satisfies the condition:

$$\tau\text{-titling infinite} \Leftrightarrow \text{wild.}$$

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Thank you very much for your attention !