τ-tilting theory and Hecke algebras

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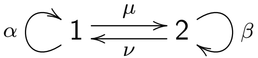
Outline

Introduction

Bound quiver algebra

$\mathcal{A} = KQ/I$, e.g.,

$Q:$ 

$$I = \langle \alpha^3, \beta^3, \mu\nu, \nu\mu, \alpha\mu\beta, \beta\nu\alpha, \nu\alpha\mu, \mu\beta\nu, \nu\alpha^2\mu, \mu\beta^2\nu \rangle.$$

Bound quiver algebra

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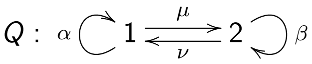

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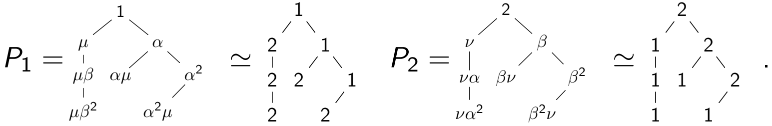
The indecomposable projective $\mathcal{A}$-modules are



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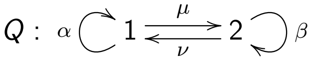

$$I = \langle \alpha^3, \beta^3, \mu\nu, \nu\mu, \alpha\mu\beta, \beta\nu\alpha, \nu\alpha\mu, \mu\beta\nu, \nu\alpha^2\mu, \mu\beta^2\nu \rangle.$$

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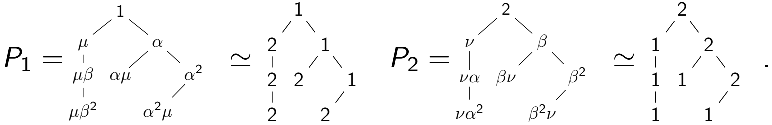
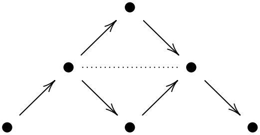


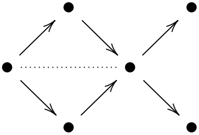
Then, $\mathcal{A} = P_1 \oplus P_2$ is naturally $\mathbb{Z}_{\geq 0}$-graded.

Abelian category

The module category of $\mathcal{B} = K(\ 1 \longrightarrow 2 \longrightarrow 3\ )$ is given by

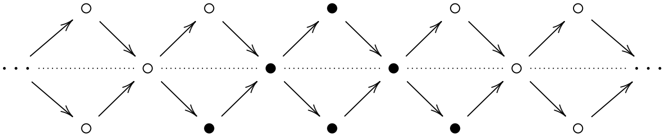


The module category of $\mathcal{C} = K(\ 1 \longrightarrow 2 \longleftarrow 3\ )$ is given by



Triangulated category

The derived category of $\mathcal{B} = K(\ 1 \longrightarrow 2 \longrightarrow 3\ )$ is given by



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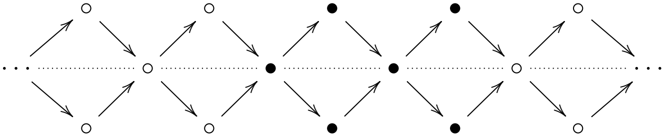

Derived equivalence

$\mathcal{A}, \mathcal{B}$: two arbitrary algebras.

Theorem (Rickard, 1989)

$\mathcal{A}$ is derived equivalent to $\mathcal{B}$, i.e., $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\ \mathcal{A}) \xrightarrow{\sim} \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\ \mathcal{B})$, if and only if there is a tilting complex T in tilt $\mathcal{A}$ such that

$$\mathcal{B} \simeq \mathsf{End}_{\mathcal{A}}\ T.$$

Hecke algebras

Coxeter generators

The Iwahori-Hecke algebra $\mathcal{H}(\mathfrak{S}_n)$ is the $\mathbb{Z}[q, q^{-1}]$-algebra generated by $\{T_i \mid 1 \leq i \leq n-1\}$ subject to

$$(T_i + 1)(T_i - q) = 0$$

$$T_i T_j = T_j T_i \text{ if } |i - j| \neq 1, \quad T_i T_j T_i = T_j T_i T_j \text{ if } |i - j| = 1.$$

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The cyclotomic Hecke algebra (a.k.a. Ariki-Koike algebra) $\mathcal{H}_n^\Lambda$ associated with $\Lambda = \Lambda_{i_1} + \Lambda_{i_2} + \cdots + \Lambda_{i_k}$ and a complex reflection of type $G(k, 1, n)$, is generated by $\{T_0\} \cup \{T_i \mid 1 \leq i \leq n-1\}$ subject to

$$(T_i + 1)(T_i - q) = 0, \quad \prod_{j=1}^{k}(T_0 - q^{i_j}) = 0$$

$$(T_0 T_1)^2 = (T_1 T_0)^2, \quad T_0 T_i = T_i T_0 \text{ if } i \geq 2,$$

$$T_i T_j = T_j T_i \text{ if } |i-j| \neq 1, \quad T_i T_j T_i = T_j T_i T_j \text{ if } |i-j| = 1.$$

Categorification

- $\mathfrak{g}$: certain Kac-Moody Lie algebra
- $V(\Lambda)$: the irreducible highest weight module w.r.t. Λ over $\mathfrak{g}$

Lie Theory	**Representation Theory**
Weight spaces of $V(\Lambda)$	Blocks of $\mathcal{H}_n^\Lambda$
Crystal graph of $V(\Lambda)$	Socle branching rule for $\mathcal{H}_n^\Lambda$
Canonical basis in $V(\Lambda)$ over $\mathbb{C}$	Indecom. projective $\mathcal{H}_n^\Lambda$-modules
Action of the Weyl group of $\mathfrak{g}$ on $V(\Lambda)$	Derived equivalences between blocks of $\mathcal{H}_n^\Lambda$

Quiver Hecke algebra

a.k.a. Khovanov-Lauda-Rouquier algebra

The quiver Hecke algebra $R(n)$ with $(Q_{i,j}(u,v))_{0\leq i,j\leq \ell}$ is the K-algebra generated by

$$\{e(\nu) \mid \nu = (\nu_1, \nu_2, \ldots, \nu_n) \in I^n\}, \quad \{x_i \mid 1 \leq i \leq n\}, \quad \{\psi_j \mid 1 \leq j \leq n-1\},$$

subject to the following relations:

The $\mathbb{Z}$-grading on $R^\Lambda(n)$ is defined by

$$\deg(e(v)) = 0, \ \deg(x_i) = 2, \ \deg(\phi_j) \in \{0, 1, 2, -2, 3\}.$$

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subject to the following relations:

(1) $e(\nu)e(\nu')=\delta_{\nu,\nu'}e(\nu),\ \sum_{\nu\in I^n} e(\nu)=1,\ x_ix_j=x_jx_i,\ x_ie(\nu)=e(\nu)x_i.$

(2) $\psi_ie(\nu)=e(s_i(\nu))\psi_i,\ \psi_i\psi_j=\psi_j\psi_i$ if $|i-j|>1,\ \psi_i^2e(\nu)=Q_{\nu_i,\nu_{i+1}}(x_i,x_{i+1})e(\nu).$

(3) $(\psi_ix_j-x_{s_i(j)}\psi_i)e(\nu)=\begin{cases} -e(\nu) & \text{if } j=i \text{ and } \nu_i=\nu_{i+1},\\ e(\nu) & \text{if } j=i+1 \text{ and } \nu_i=\nu_{i+1},\\ 0 & \text{otherwise.}\end{cases}$

(4) $(\psi_{i+1}\psi_i\psi_{i+1}-\psi_i\psi_{i+1}\psi_i)e(\nu)=\begin{cases} \frac{Q_{\nu_i,\nu_{i+1}}(x_i,x_{i+1})-Q_{\nu_i,\nu_{i+1}}(x_{i+2},x_{i+1})}{x_i-x_{i+2}}e(\nu) & \text{if } \nu_i=\nu_{i+2},\\ 0 & \text{otherwise.}\end{cases}$

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$$\deg(e(v))=0,\ \deg(x_i)=2,\ \deg(\phi_j)\in\{0,1,2,-2,3\}.$$

Cyclotomic quiver Hecke algebra

The cyclotomic quiver Hecke algebra $R^\Lambda(n)$ is defined as the quotient of $R(n)$ modulo the relation

$$x_1^{\langle h_{\nu_1}, \Lambda \rangle} e(\nu) = 0.$$

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Theorem (Brundan-Kleshchev, 2009)

We have $H_n^{\Lambda} \simeq R^{\Lambda}(n)$ if

$$Q_{i,i}(u,v) = 0, Q_{i,j}(u,v) = Q_{j,i}(v,u),$$

and $Q_{0,1}(u,v) = u^2 - 2uv + v^2$ if $\ell = 1$, if $\ell \geq 2$,

$$\begin{aligned} Q_{\ell,0}(u,v) &= u + (-1)^{\ell+1} \cdot v, \\ Q_{i,i+1}(u,v) &= u + v \qquad \text{if } 0 \leq i < \ell, \\ Q_{i,j}(u,v) &= 1 \qquad\quad\;\; \text{if } j \not\equiv_{\ell+1} i, i \pm 1. \end{aligned}$$

Properties of $R^\Lambda(n)$

- $R^\Lambda(n)$ categorifies the quantum group $U_q(\mathfrak{g})$.
- $R^\Lambda(n)$ is a symmetric algebra, see [Shan-Varagnolo-Vasserot, 2017].
- In affine type $\mathbb{A}$ ([Brundan-kleshchev, 2009]), we have,

$$\dim_q e(v)R^\Lambda(n)e(v') = \sum_{\substack{\lambda\vdash n,\ S,T\in \mathrm{Std}(\lambda),\\ \omega_S=\nu,\ \omega_T=\nu'}} q^{\deg(S)+\deg(T)}$$

- $R^\Lambda(\beta) \sim_{\text{derived}} R^\Lambda(\beta')$ if both $\Lambda-\beta$ and $\Lambda-\beta'$ lie in

$$\{\mu - m\delta \mid \mu \in \max^+(\Lambda), m\in \mathbb{Z}_{\geq 0}\},$$

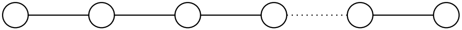
which is the W-orbit of the set $P(\Lambda)$ of weights of $V(\Lambda)$, where W is the affine symmetric group and $V(\Lambda)$ is the integrable highest weight module of the quantum group. See: [Chuang-Rouquier, 2008].

Main result on Hecke algebras

$\mathcal{A}$: a non-local block algebra of $R^{\Lambda}(n)$ of affine type $\mathbb{A}_{\ell}^{(1)}$.

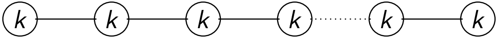
Theorem (Ariki-Song-W., 2023)

(1) If $\mathcal{A}$ is representation-finite, it is derived equivalent to a Brauer tree algebra given by



(2) If $\mathcal{A}$ is tame, it is derived equivalent to one of

- the Brauer graph algebra given by



where $k = |\Lambda|$ and $\#$vertices $= \ell + 1$.

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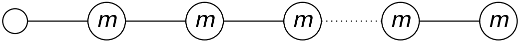
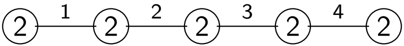


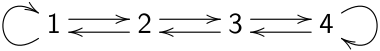
where m and $\#$vertices could be calculated explicitly.

Brauer graph algebra

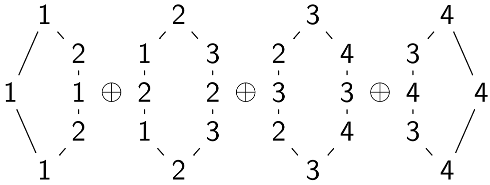
e.g., the Brauer graph



gives a bound quiver algebra with



and



τ-tilting theory

Tilting complex

Set $\mathcal{K}_\mathcal{A} := \mathsf{K}^\mathsf{b}(\operatorname{proj} \mathcal{A})$. We fix

- thick T: the smallest thick subcategory of $\mathcal{K}_\mathcal{A}$ containing T

Definition (Aihara-Iyama, 2012)

A complex $T \in \mathcal{K}_\mathcal{A}$ is said to be

(1) presilting if $\operatorname{Hom}_{\mathcal{K}_\mathcal{A}}(T, T[i]) = 0$, for any $i > 0$.

(2) silting if T is presilting and thick $T = \mathcal{K}_\mathcal{A}$.

(3) tilting if T is silting and $\operatorname{Hom}_{\mathcal{K}_\mathcal{A}}(T, T[i]) = 0$, for any $i < 0$.

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Silting $=$ **Tilting** for symmetric algebras, e.g., $R^\Lambda(n)$.

Silting mutation

Definition (Aihara-Iyama, 2012)

$\forall T, S \in \operatorname{silt} \mathcal{A}$, we say $T \geq S$ if $\operatorname{Hom}_{\mathcal{K}_{\mathcal{A}}}(T, S[i]) = 0$ for any $i > 0$.

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Theorem-Definition (Aihara-Iyama, 2012)

For any S, $T \in \operatorname{silt} \mathcal{A}$, the following conditions are equivalent.

(1) S is a irreducible left mutation of T, i.e., $S = \mu_i^-(T)$.

(2) T is a irreducible right mutation of S, i.e., $T = \mu_i^+(S)$.

(3) $T > S$ and there is no $X \in \operatorname{silt} \mathcal{A}$ such that $T > X > S$.

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Mutation graph

$$T_1 \oplus \cdots \oplus T_j \oplus \cdots \oplus T_n \longrightarrow T_1 \oplus \cdots \oplus T_j^* \oplus \cdots \oplus T_n$$

2-term silting complex

A complex in $\mathcal{K}_{\mathcal{A}}$ is called 2-term if it is homotopy equivalent to a complex T of the form

$$\cdots \longrightarrow 0 \longrightarrow T^{-1} \xrightarrow{d_T^{-1}} T^0 \longrightarrow 0 \longrightarrow \cdots .$$

Theorem (Brustle-Yang, 2013)

For any $T \in \operatorname{silt} \mathcal{A}$, set $\mathcal{B} := \operatorname{End}_{\mathcal{A}} T$. There exists a bijection

$$\{S \mid T \geq S \geq T[1]\} \xrightarrow{1:1} \text{2-silt}\,\mathcal{B}$$

which commutes with mutations.

τ-tilting theory

Theorem (Adachi-Iyama-Reiten, 2014)

There exists a bijection between 2-silt $\mathcal{A}$ and sτ-tilt $\mathcal{A}$ given by

$$T \mapsto H^0(T),$$

which commutes with mutations.

Definition (AIR, 2014)

Let M be a right $\mathcal{A}$-module. Then,

(1) M is called τ-rigid if $\mathrm{Hom}_{\mathcal{A}}(M, \tau M) = 0$.

(2) M is called τ-tilting if M is τ-rigid and $|M| = |\mathcal{A}|$.

(3) M is called support τ-tilting if M is a τ-tilting $(\mathcal{A}/\mathcal{A}e\mathcal{A})$-module for an idempotent e of $\mathcal{A}$.

Nakayama functor $\nu(-)$: proj $A \to$ inj A

Take the minimal projective presentation of M as

$$P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0,$$

the Auslander-Reiten translation τM is defined by the following exact sequence:

$$0 \longrightarrow \tau M \longrightarrow \nu P_1 \xrightarrow{\nu f_1} \nu P_0,$$

that is, $\tau M = \ker \nu f_1$.

Mutation graph

Proposition (Adachi-Iyama-Reiten, 2014)

If the mutation graph $\mathcal{H}(\mathsf{s}\tau\text{-tilt}\,\mathcal{A})$ contains a finite connected component Δ, then $\mathcal{H}(\mathsf{s}\tau\text{-tilt}\,\mathcal{A}) = \Delta$.

Mutation graph

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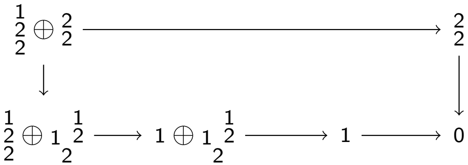
e.g., set $\mathcal{A} := KQ/I$ with

$$1 \xrightarrow{\alpha} 2 \circlearrowright \beta \quad \text{and} \quad \beta^2 = 0.$$

The mutation graph $\mathcal{H}(\mathsf{s}\tau\text{-tilt}\,\mathcal{A})$ is displayed as



Main result on symmetric algebras

Theorem (Ariki-Song-Hudak-W., 2024)

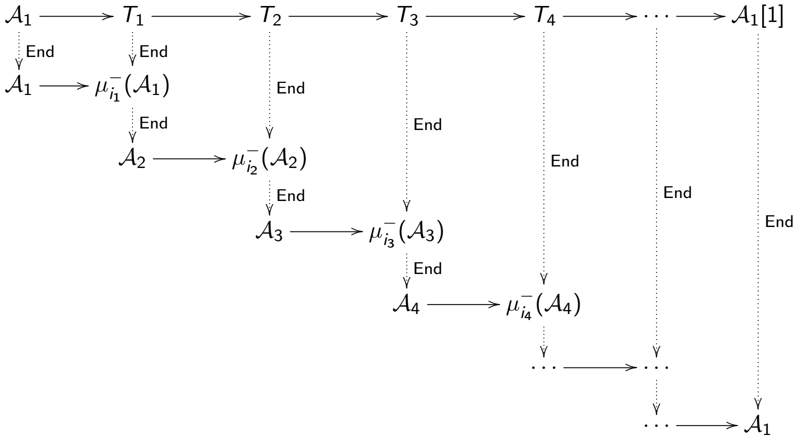
Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_s$ be symmetric algebras which are derived equivalent to each other. If

(1) the set sτ-tilt $\mathcal{A}_i$ is finite, for any $1 \leq i \leq s$.

(2) $\mathrm{End}_{\mathcal{A}_i}(\mu_X^-(\mathcal{A}_i)) \simeq \mathcal{A}_j$ for some $1 \leq j \leq s$, with respect to any indecomposable projective direct summand X of the left regular module $\mathcal{A}_i$ for $1 \leq i \leq s$,

then any algebra $\mathcal{B}$ derived equivalent to $\mathcal{A}_1$, is Morita equivalent to $\mathcal{A}_i$ for some $1 \leq i \leq s$.

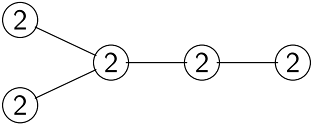
Proof strategy: Suppose $T_i \in$ 2-tilt $\mathcal{A}_1$, and $\mathcal{A}_i := \mathrm{End}_{\mathcal{K}_{\mathcal{A}_i}} T_{i-1}$.



Example of affine type $\mathbb{A}$

e.g., the Brauer graph algebra given by

is derived equivalent to Brauer graph algebras given by



and

Let $\mathcal{A}$ be a Brauer graph algebra with Brauer graph $\Gamma_{\mathcal{A}}$.

Theorem (Antipov-Zvonareva, 2022)

If $\mathcal{B}$ is derived equivalent to $\mathcal{A}$, $\mathcal{B}$ is also a Brauer graph algebra.

Theorem (Opper-Zvonareva, 2022)

$\mathcal{A} \sim_{\text{derived}} \mathcal{B}$ if and only if the following conditions hold.

(1) $\Gamma_{\mathcal{A}}$ and $\Gamma_{\mathcal{B}}$ share the same number of vertices, edges, faces,

(2) the multisets of multiplicities and the multisets of perimeters of faces of $\Gamma_{\mathcal{A}}$ and $\Gamma_{\mathcal{B}}$ coincide,

(3) either both or none of $\Gamma_{\mathcal{A}}$ and $\Gamma_{\mathcal{B}}$ are bipartite.

Application

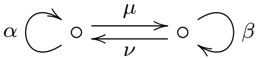
Example of affine type $\mathbb{C}$

$\mathcal{A}$: the block algebra of $R^{\Lambda}(n)$ of affine type $\mathbb{C}_{\ell}^{(1)}$ with respect to

$$\Lambda = \Lambda_0 + 2\Lambda_1, \quad \beta = \alpha_0 + \alpha_1.$$

Proposition (Ariki-Hudak-Song-W., 2024)

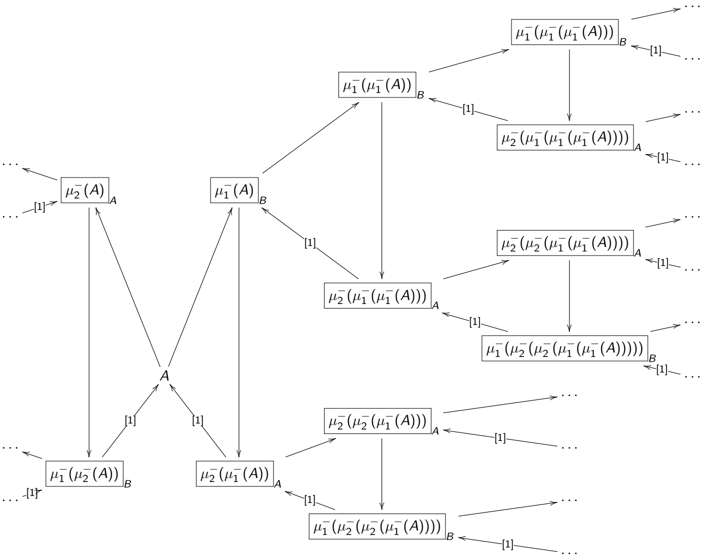
$\mathcal{A}$ is tame and Morita equivalent to $A = KQ/I$ with



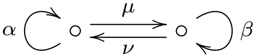
bounded by $\alpha^2 = 0, \beta^2 = \nu\mu, \alpha\mu = \mu\beta, \beta\nu = \nu\alpha$.

This is not a Brauer graph algebra!

Tilting quiver of A



It gives $B = KQ/I$ with

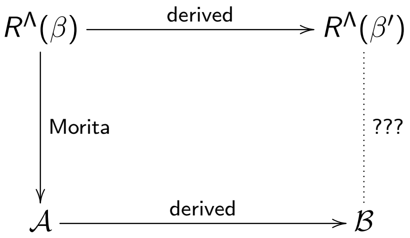


and $\alpha^2 = \mu\nu, \beta^2 = \nu\mu, \alpha\mu = \mu\beta, \beta\nu = \nu\alpha,\ \mu\nu\mu = \nu\mu\nu = 0$.

Proposition (Ariki-Hudak-Song-W., 2024)

If C is derived equivalent to A, then C is isomorphic to A or B.

Open Problem



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Thank you for listening!