Brick finiteness of classical Schur algebras

Qi WANG
Yau Mathematical Sciences Center
Tsinghua University

2025年同调与表示论学术论坛
合肥工业大学
2025年4月19日

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$$\begin{aligned} S(n,r) &= \mathrm{End}_{\mathbb{C}\mathfrak{S}_r}\left((\mathbb{C}^n)^{\otimes r}\right) \\ &= \mathrm{Image}\{\mathbb{C}\mathrm{GL}_n \to \mathrm{End}_{\mathbb{C}}\left((\mathbb{C}^n)^{\otimes r}\right)\} \end{aligned}$$

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- a quasi-hereditary, cellular algebra.

Brick finiteness

Let $A = \mathbb{F}Q/I$. A module N is called a **brick** if $\mathrm{End}_A(N) \simeq \mathbb{F}$.

Then, A is said to be

(1) (Chindris-Kinser-Weyman, 2012) **Schur-representation-finite** if there are finitely many bricks of a fixed dimension.

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- (2) $\Rightarrow$ (1) is obvious.
- (1) $\Rightarrow$ (2) is not verified; no counterexample.

Brick filtration

For any $M \in \operatorname{mod} A$, we give a filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{n-1} \subset M_n = M.$$

Let N be a brick. We set

$$\mathcal{E}(N) := \{N' \mid N'_i / N'_{i-1} = N\}.$$

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Any $M \in \operatorname{mod} A$ admits a filtration with $M_i/M_{i-1} \in \mathcal{E}(N^{(i)})$ s.t.,

- $N^{(1)}, N^{(2)}, \cdots, N^{(n)}$ is a sequence of bricks;
- $\operatorname{Hom}_A(N^{(i)}, N^{(j)}) = 0$ if $i < j$.

An algebra A is brick finite, if and only if,

- the number of torsion classes in mod A is finite;
- the number of wide subcategories of mod A is finite;
- the number of τ-tilting modules in mod A is finite;
- the number of 2-term silting complexes in $\mathrm{D}^{\mathrm{b}}(A)$;
- the number of intermediate algebraic t-structures in $\mathrm{D}^{\mathrm{b}}(A)$;
- etc.

See [Adachi-Iyama-Reiten 2014], [Iyama-Jorgensen-Yang 2014], [Demonet-Iyama-Jasso 2019], [Ringel 2024] for references.

Schur algebra

Let $\mathbb{F}$ be an algebraically closed field, char $\mathbb{F} = 0$ or p (a prime). Replace $\mathbb{C}^n$ by a an n-dimensional vector space V over $\mathbb{F}$ with a basis $\{v_1, v_2, \ldots, v_n\}$. Then,

$$S(n,r) := \mathrm{End}_{\mathbb{F}\mathfrak{S}_r}(V^{\otimes r})$$

with

$$(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r}) \cdot \sigma = v_{i_{\sigma(1)}} \otimes v_{i_{\sigma(2)}} \otimes \cdots \otimes v_{i_{\sigma(r)}}.$$

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- $S(n,r)$ is not necessarily basic;
- $S(n,r)$ is not necessarily indecomposable;

We denote by $\overline{S(n,r)}$ the basic algebra of $S(n,r)$.

Let $\Omega(n,r) := \{\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \mid \lambda : \text{ partition of } r\}$.

(1) Cellular algebra (or quasi-hereditary algebra with a duality):

$$\overline{S(n,r)} = \mathrm{End}_{S(n,r)}\left(\bigoplus_{\lambda\in\Omega(n,r)} P(\lambda)\right)$$

- $L(\lambda)$: simple module, $\Delta(\lambda)$: Weyl module
- $P(\lambda)$: indecomposable projective module

We need $[P(\lambda) : \Delta(\mu)]$ and $[\Delta(\lambda) : L(\mu)]$.

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(2) Endomorphism algebra of Young modules:

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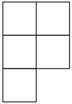
Specht module

Let $[\lambda]$ be the Young diagram of a partition λ. We call λ *p*-**regular** if no *p* rows of $[\lambda]$ have the same length, and *p*-**singular** otherwise.
e.g.,

$[(2,2,1)] =$  is $\begin{cases} \text{2-singular} \\ \text{3-regular} \end{cases}$

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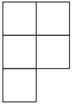
$[(2,2,1)] =$  is $\begin{cases} \text{2-singular} \\ \text{3-regular} \end{cases}$

Let S^λ be the **Specht module** of $\mathbb{F}\mathfrak{G}_r$ corresponding to λ.

- If $\operatorname{char}\mathbb{F} = 0$, then $\left\{ S^\lambda \mid \lambda : \text{ partition of } r \right\}$ is a complete set of pairwise non-isomorphic simple $\mathbb{F}\mathfrak{G}_r$-modules.
- If $\operatorname{char}\mathbb{F} = p$, then $\left\{ D^\lambda \mid \lambda : \text{ } p\text{-regular partition of } r \right\}$ is a complete set of pairwise non-isomorphic simple $\mathbb{F}\mathfrak{G}_r$-modules, where $D^\lambda := S^\lambda / (\operatorname{rad} S^\lambda)$.

According to [James, 1978], the **decomposition matrix** $[S^\lambda : D^\mu]$ of $\mathbb{F}\mathfrak{S}_r$ over $\operatorname{char}\mathbb{F} = p$ has the form

$$\begin{array}{r}
S^\lambda, \lambda \ p\text{-regular} \left\{ \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \right. \\
\\
S^\lambda, \lambda \ p\text{-singular} \left\{ \begin{array}{c} \\ \\ \end{array} \right.
\end{array}
\overbrace{\begin{pmatrix}
1 & & & & \\
* & 1 & & & \\
* & * & 1 & & \\
\vdots & \vdots & \vdots & \ddots & \\
* & * & * & \dots & 1 \\
- & - & - & - & - \\
* & * & * & \dots & * \\
* & * & * & \dots & *
\end{pmatrix}}^{D^\mu, \mu \ p\text{-regular}}.$$

Permutation module

We have

$$V^{\otimes r} \simeq \bigoplus_{\lambda \in \Omega(n,r)} n_\lambda M^\lambda$$

with multiplicities n_λ, where

- $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ is a partition of r with at most n parts;
- $\mathfrak{G}_\lambda := \mathfrak{G}_{\lambda_1} \times \mathfrak{G}_{\lambda_2} \times \ldots \times \mathfrak{G}_{\lambda_n}$ is the Young subgroup of $\mathfrak{G}_r$;
- M^λ is the induced $\mathbb{F}\mathfrak{G}_r$-module $1_{\mathfrak{G}_\lambda} \uparrow^{\mathfrak{G}_r}$.

Remark: M^λ is not necessarily indecomposable.

Young module

A permutation module M^λ is liftable by a p-modular system.

- χ^λ: the ordinary irreducible character of $\mathfrak{S}_r$ corresponding to λ over $\operatorname{char}\mathbb{F} = 0$. ($\overset{1:1}{\longleftrightarrow}$ Specht module S^λ)
- $\operatorname{char} M^\lambda$: the associated character of M^λ over $\operatorname{char}\mathbb{F} = p$.

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We have

$$\operatorname{char} M^\lambda = \chi^\lambda + \sum_{\mu \rhd \lambda} k_\mu \chi^\mu$$

$$\Rightarrow \qquad M^\lambda := Y^\lambda \oplus \bigoplus_{\mu \rhd \lambda} (Y^\mu)^{k_\mu}$$

Here, Y^λ is the **Young module** corresponding to λ.

Theorem (Martin, 1993)

Let Y be an indecomposable direct summand of a permutation module M^λ for any $\lambda \in \Omega(n,r)$. Then, $Y \in \{ Y^\lambda \mid \lambda \in \Omega(n,r) \}$.

We have

$$\overline{S(n,r)} := \mathrm{End}_{\mathbb{F}\mathfrak{G}_r}\left(\bigoplus_{\lambda \in \Omega(n,r)} Y^\lambda \right).$$

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There are some properties of Young modules.

- Y^λ is self-dual, i.e., $Y^\lambda = \mathrm{Hom}(Y^\lambda, \mathbb{F})$;
- Y^λ admits a Specht filtration;
- each composition factor D^μ of Y^λ is given by a partition μ with at most n parts.

Example

Let $p = 2$, $n = 2$ and $r = 11$. There are two blocks of $\mathbb{F}\mathfrak{S}_{11}$:

$$\begin{matrix}(11)\\(9,2)\\(7,4)\end{matrix}\begin{pmatrix}1 & & \\ 0 & 1 & \\ 1 & 0 & 1\end{pmatrix} \qquad \begin{matrix}(10,1)\\(8,3)\\(6,5)\end{matrix}\begin{pmatrix}1 & & \\ 1 & 1 & \\ 0 & 1 & 1\end{pmatrix}$$

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B_1: 2-core (1) $\qquad$ B_2: 2-core $(2,1)$

According to [Henke, 1999], we have

$$\begin{aligned} \mathrm{char}\, Y^{(10,1)} &= \chi^{(10,1)}, \\ \mathrm{char}\, Y^{(8,3)} &= \chi^{(10,1)} + \chi^{(8,3)}, \\ \mathrm{char}\, Y^{(6,5)} &= \chi^{(10,1)} + \chi^{(8,3)} + \chi^{(6,5)}. \end{aligned}$$

We have

$$Y^{(10,1)} = D^{(10,1)} \qquad Y^{(8,3)} = \begin{array}{c} D^{(10,1)} \\ D^{(8,3)} \\ D^{(10,1)} \end{array} \qquad Y^{(6,5)} = \begin{array}{ccccccc} & D^{(8,3)} & & & & D^{(10,1)} \\ / & & \backslash & & / & \\ D^{(10,1)} & & & D^{(6,5)} & & \\ & & & \backslash & & \\ & & & & D^{(8,3)} & \end{array}$$

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Then, $S_{B_2} := \mathrm{End}_{\mathbb{F}\mathfrak{S}_{11}}(Y^{(10,1)} \oplus Y^{(8,3)} \oplus Y^{(6,5)}) \simeq \mathbb{F}Q/I$ with

$$Q: \ (10,1) \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} (6,5) \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} (8,3)$$

$$I: \langle \alpha_1\beta_1, \beta_2\alpha_2, \alpha_1\alpha_2\beta_2, \alpha_2\beta_2\beta_1 \rangle.$$

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Similarly, $S_{B_1} := \mathbb{F}\left((11) \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} (7,4) \qquad (9,2) \right) \Big/ \langle \alpha_1\beta_1 \rangle.$

We conclude that $\overline{S(2,11)} = S_{B_1} \oplus S_{B_2}$.

Representation type

Theorem ([Erdmann, 93], [Xi, 93], [Doty-Nakano, 98], [Doty-Erdmann-Martin-Nakano, 99])

Let $\operatorname{char}\mathbb{F} = 0$ or p. Then, $S(n, r)$ is

- semi-simple iff $\operatorname{char}\mathbb{F} = 0$ or $p > r$ or $p = 2,\ n = 2,\ r = 3$;
- representation-finite iff $p = 2,\ n = 2,\ r = 5, 7$ or $p \geqslant 2,\ n = 2,\ r < p^2$ or $p \geqslant 2,\ n \geqslant 3,\ r < 2p$;
- tame iff $p = 2,\ n = 2,\ r = 4, 9, 11$ or $p = 3,\ n = 2,\ r = 9, 10, 11$ or $p = 3,\ n = 3,\ r = 7, 8$.

Otherwise, $S(n, r)$ is wild.

On Hecke algebra

Suppose $n \geq r$ and char $\mathbb{F} = 0$. $S(n,r)$ is a graded algebra, s.t.

$$[S^\lambda, D^\mu]_v = \sum_{i \geq 0} [\mathrm{rad}^i(\Delta(\lambda^t))/\mathrm{rad}^{i+1}(\Delta(\lambda^t)) : L(\mu^t)] \cdot v^i.$$

where λ^t is the transposed partition of λ, see [Shan, 2010]. In this case, $\mathbb{F}\mathfrak{G}_r = eS(n,r)e$ for some idemponent e of $S(n,r)$.

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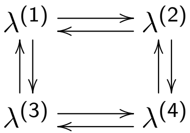
Let $\lambda^{(i)}$ be a p-regular partition, for $i \in \{1,2,3,4\}$. Then,

$$[S^\lambda, D^\mu]_v : \begin{pmatrix} 1 & & & \\ v & 1 & & \\ v & 0 & 1 & \\ v^2 & v & v & 1 \end{pmatrix} \quad \Rightarrow$$



τ-tilting theory

Auslander-Reiten translation

Nakayama functor $\nu(-)$: proj $A \to$ inj A

Let M be an A-module with a minimal projective presentation

$$P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0,$$

the **Auslander-Reiten translation** τM is defined by the following exact sequence

$$0 \longrightarrow \tau M \longrightarrow \nu P_1 \xrightarrow{\nu f_1} \nu P_0,$$

that is, $\tau M = \ker\ \nu f_1$.

Definition (Adachi-Iyama-Reiten, 2014)

Let M be a right A-module. Then,

(1) M is called τ-rigid if $\mathrm{Hom}_A(M, \tau M) = 0$.

(2) M is called τ-tilting if M is τ-rigid and $|M| = |A|$.

(3) M is called support τ-tilting if M is a τ-tilting (A/AeA)-module for an idempotent e of A.

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We have

$$\mathrm{i}\tau\text{-rigid}\, A \quad \text{gives} \quad \tau\text{-tilt}\, A \subseteq \mathrm{s}\tau\text{-tilt}\, A \subseteq \tau\text{-rigid}\, A$$

Mutation

Reminder: $M_1 \oplus \cdots \oplus M_j \oplus \cdots \oplus M_n \Rightarrow M_1 \oplus \cdots \oplus M_j^* \oplus \cdots \oplus M_n$.

- add(M): the full subcategory whose objects are direct summands of finite direct sums of copies of M;
- Fac(M): the full subcategory whose objects are factor modules of finite direct sums of copies of M.

Definition (Adachi-Iyama-Reiten, 2014)

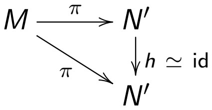
Let $M = M_1 \oplus \cdots \oplus M_j \oplus \cdots \oplus M_n$ with $M_j \notin \mathrm{Fac}(M/M_j)$. Take a minimal left add(M/M_j)-approximation π with an exact sequence

$$M_j \xrightarrow{\pi} Z \longrightarrow \mathrm{coker}\ \pi \longrightarrow 0.$$

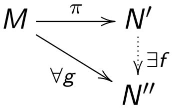
We call $\mu_j^-(M) := \mathrm{coker}\ \pi \oplus (M/M_j)$ the left mutation of M with respect to M_j, which is again a support τ-tilting A-module.

A morphism $\pi : M \to N'$ is a **minimal left** add(N)**-approximation of** M if $N' \in \text{add}(N)$ and it satisfies:

(1) any $h : N' \to N'$ satisfying $h \circ \pi = \pi$ is an automorphism.



(2) for any $N'' \in \text{add}(N)$ and $g : M \to N''$, there exists $f : N' \to N''$ such that $f \circ \pi = g$.



Mutation quiver

We draw an arrow $M \to \mu_j^-(M)$, it gives a quiver $\mathcal{Q}(\mathsf{s}\tau\text{-tilt}\, A)$. e.g., set $\Lambda := \mathbb{F}Q/I$ with

$$Q:\ 1 \xrightarrow{\alpha} 2 \circlearrowleft \beta \qquad \text{and} \qquad I: \langle \beta^2 \rangle,$$

the quiver $\mathcal{Q}(\mathsf{s}\tau\text{-tilt}\, \Lambda)$ is displayed as

$$\begin{array}{ccccccc}
\begin{smallmatrix}1\\2\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\2\end{smallmatrix} & & \longrightarrow & & & & \begin{smallmatrix}2\\2\end{smallmatrix} \\
\downarrow & & & & & & \downarrow \\
\begin{smallmatrix}1\\2\\2\end{smallmatrix} \oplus \begin{smallmatrix} & & 1\\ 1 & & 2 \\ & 2 & \end{smallmatrix} & \longrightarrow & 1 \oplus \begin{smallmatrix} & & 1\\ 1 & & 2 \\ & 2 & \end{smallmatrix} & \longrightarrow & 1 & \longrightarrow & 0
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\end{array}$$

Proposition (Adachi-Iyama-Reiten, 2014)

If $\mathcal{Q}(\mathrm{s}\tau\text{-tilt}\, A)$ contains a finite connected component Δ, then $\mathcal{Q}(\mathrm{s}\tau\text{-tilt}\, A) = \Delta$.

Connection with bricks

Let fbrick A be the set of bricks M such that the smallest torsion class $\mathsf{T}(M)$ containing M is functorially finite.

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Theorem (Demonet-Iyama-Jasso, 2019)

There exists a bijection between iτ-rigid A and fbrick A, given by

$$X \mapsto X/\mathrm{rad}_B(X),$$

where $B := \mathrm{End}_A(X)$. If iτ-rigid A is finite, fbrick $A =$ brick A.

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Theorem (Demonet-Iyama-Jasso, 2019)

There exists a bijection between iτ-rigid A and fbrick A, given by

$$X \mapsto X/\mathrm{rad}_B(X),$$

where $B := \mathrm{End}_A(X)$. If iτ-rigid A is finite, fbrick A = brick A.

e.g.,

iτ-rigid Λ	$\begin{smallmatrix}1\\2\\2\end{smallmatrix}$	$\begin{smallmatrix}2\\2\end{smallmatrix}$	$\begin{smallmatrix} & & 1\\ 1 & & 2\\ & 2 & \end{smallmatrix}$	1
brick Λ	$\begin{smallmatrix}1\\2\\2\end{smallmatrix}$	2	$\begin{smallmatrix}1\\2\end{smallmatrix}$	1

Useful results

Proposition (Eisele-Janssens-Raedschelders, 2018)

Let I be a two-sided ideal generated by central elements in rad A. Then,

$$\mathrm{s}\tau\text{-tilt}\, A \simeq \mathrm{s}\tau\text{-tilt}\,(A/I).$$

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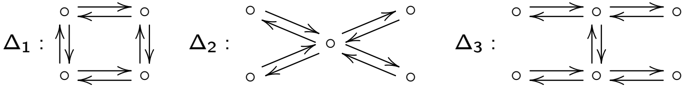
Proposition ([Adachi, 2016], [Mousavand, 2019])

A path algebra $\mathbb{F}Q$ of type $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_n$, is minimal brick infinite.

e.g., minimal brick infinite quivers:



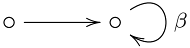
Others

The brick finiteness is known, for example, for

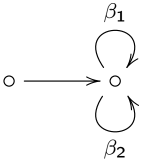
- preprojective algebras of Dynkin type (Mizuno, 2014);
- algebras with radical square zero (Adachi, 2016);
- Brauer graph algebras (Adachi-Aihara-Chan, 2018);
- gentle algebras (Plamondon, 2018);
- simply connected algebras (W., 2019)
- 0-Hecke algebras and 0-Schur algebras (Miyamoto-W., 2022)
- Hecke algebras of type A over $e \geq 3$ (Ariki-Lyle-Speyer, 2022)
- Borel-Schur algebras (W., 2023)

Proposition (W., 2022)

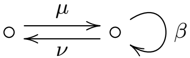
Let A be one of the following algebras,

(1) 

with $\beta^4 = 0$,

(2) 

with $\beta_1^2 = \beta_2^2 = \beta_1\beta_2 = \beta_2\beta_1 = 0$,

(3) 

with $\beta^3 = \beta\nu = \nu\mu\nu = \nu\mu\beta^2 = 0$.

Then, A is a minimal brick infinite algebra.

Main result

Representation-finite

Proposition ([Erdmann, 93], [Donkin-Reiten, 94])

Let A be a representation-finite block of $S(n, r)$. Then, it is Morita equivalent to $\mathcal{A}_m := \mathbb{F}Q/I$ with

$$Q: \ 1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \cdots \underset{\beta_{m-2}}{\overset{\alpha_{m-2}}{\rightleftarrows}} m-1 \underset{\beta_{m-1}}{\overset{\alpha_{m-1}}{\rightleftarrows}} m \ ,$$

$$I: \langle \alpha_1\beta_1, \alpha_i\alpha_{i+1}, \beta_{i+1}\beta_i, \beta_i\alpha_i - \alpha_{i+1}\beta_{i+1} \mid 1 \leq i \leq m-2 \rangle.$$

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$$I: \langle \alpha_1\beta_1, \alpha_i\alpha_{i+1}, \beta_{i+1}\beta_i, \beta_i\alpha_i - \alpha_{i+1}\beta_{i+1} \mid 1 \le i \le m-2 \rangle.$$

Proposition ([Asashiba-Mizuno-Nakashim, 20], [Aoki, 21])

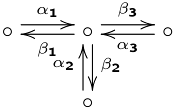
We have $\#\mathrm{s}\tau\text{-tilt}\,\mathcal{A}_m = \binom{2m}{m}$.

Proof: $\mathcal{A}_m$ is a quotient of a Brauer tree algebra modulo the ideal generated by the central element $\alpha_1\beta_1$.

Tame

Proposition (Doty-Erdmann-Martin-Nakano, 1999)

Let A be a tame block of **tame** Schur algebras. Then, it is Morita equivalent to one of the following algebras:

- $\mathcal{D}_3$: $\circ \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \circ \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \circ$ with $\langle \alpha_1\beta_1, \beta_2\alpha_2, \alpha_1\alpha_2\beta_2, \alpha_2\beta_2\beta_1 \rangle$
- $\mathcal{D}_4$: 

with $\left\langle \begin{matrix} \alpha_1\beta_1, \alpha_2\beta_2, \alpha_3\beta_1, \alpha_3\beta_2, \alpha_1\beta_3, \alpha_2\beta_3, \\ \alpha_1\beta_2\alpha_2, \beta_2\alpha_2\beta_1, \beta_2\alpha_2 - \beta_3\alpha_3 \end{matrix} \right\rangle$
- $\mathcal{H}_4$: $-$ with $\left\langle \begin{matrix} \alpha_1\beta_1, \alpha_1\beta_2, \alpha_2\beta_1, \alpha_2\beta_2, \alpha_1\alpha_3, \\ \beta_3\beta_1, \alpha_3\beta_3 - \beta_1\alpha_1 - \beta_2\alpha_2 \end{matrix} \right\rangle$
- $\mathcal{R}_4$: $\circ \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \circ \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \circ \underset{\beta_3}{\overset{\alpha_3}{\rightleftarrows}} \circ$ with $\left\langle \begin{matrix} \alpha_1\beta_1, \alpha_1\alpha_2, \beta_2\beta_1, \\ \alpha_2\beta_2 - \beta_1\alpha_1, \alpha_3\beta_3 - \beta_2\alpha_2 \end{matrix} \right\rangle$

Theorem (W, 2020)

If $S(n, r)$ is tame, then it is brick finite.

Theorem (W, 2020)

If $S(n,r)$ is tame, then it is brick finite.

Proof: We have

A	$\mathcal{D}_3$	$\mathcal{D}_4$	$\mathcal{H}_4$	$\mathcal{R}_4$
$\#\text{s}\tau\text{-tilt}\, A$	28	114	96	88

.

These are all the blocks of tame Schur algebras. e.g., set $p = 3$,

- $\overline{S(2,9)} \simeq \mathcal{D}_4 \oplus \mathbb{F}$
- $\overline{S(2,10)} \simeq \mathcal{D}_4 \oplus \mathbb{F} \oplus \mathbb{F}$
- $\overline{S(2,11)} \simeq \mathcal{D}_4 \oplus \mathcal{A}_2$
- $\overline{S(3,7)} \simeq \mathcal{R}_4 \oplus \mathcal{A}_2 \oplus \mathcal{A}_2$
- $\overline{S(3,8)} \simeq \mathcal{R}_4 \oplus \mathcal{H}_4 \oplus \mathcal{A}_2$

Wild

Lemma 1

If $S(n, r)$ is brick infinite, then so is $S(N, r)$, for any $N > n$.

Proof: $S(n, r)$ is an idempotent truncation of $S(N, r)$. Then, see [Demonet-Iyama-Jasso, 2017].

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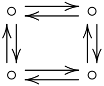
Lemma 2

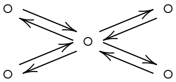
If $S(n, r)$ is brick infinite, then so is $S(n, n + r)$.

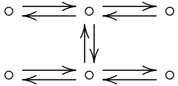
Proof: It is shown by [Erdmann, 1993] that $S(n, r)$ is a quotient of $S(n, n + r)$. Then, see [Demonet-Iyama-Reading-Reiten-Thomas, 2018].

Lemma 3

Let $A := \mathbb{F}\Delta_i/I$ for any admissible ideal I. Then, A is brick infinite.

Δ_1 :


Δ_2 :


Δ_3 :


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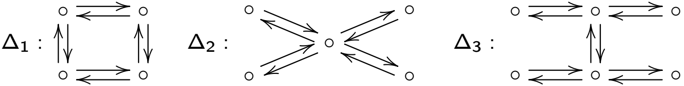

We recall

$$\Delta_4 := \mathbb{F}\left(\alpha \circlearrowright \circ \longrightarrow \circ \longleftarrow \circ \circlearrowleft \beta \right) /\langle \alpha^2, \beta^2 \rangle.$$

This is a brick infinite gentle algebra, see [Plamondon, 2018].

$\operatorname{char}\mathbb{F} = 2$

$n \backslash r$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	$\cdots$
2	S	F	S	T	F	W	F	W	T	W	T	W	W	W	W	W	W	W	W	W	W	$\cdots$

$n \backslash r$	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
3	S	F	F	W	W	W	W	W	W	W	W	W	W	$\cdots$
4	S	F	F	W	W	W	W	W	W	W	W	W	W	$\cdots$
5	S	F	F	W	W	W	W	W	W	W	W	W	W	$\cdots$
6	S	F	F	W	W	W	W	W	W	W	W	W	W	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

- $\overline{S(2,19)} \simeq \overline{S(2,8)} \oplus \mathcal{D}_3 \oplus \mathbb{F} \oplus \mathbb{F}$ is brick finite
- $\overline{S(2,10)}$ and $\overline{S(2,21)}$ are brick infinite ($\Leftarrow \Delta_1$)
- $\overline{S(3,6)}$ and $\overline{S(3,7)}$ are brick infinite ($\Leftarrow \Delta_1$)
- $\overline{S(3,8)}$ is brick infinite ($\Leftarrow \Delta_2$).
- $\overline{S(4,4)}$ is brick infinite ($\Leftarrow \Delta_1$) $\rightsquigarrow$ Hecke algebras
- $\overline{S(5,5)}$ is brick infinite ($\Leftarrow \Delta_4$) $\rightsquigarrow$ Hecke algebras

$\mathrm{char}\,\mathbb{F} = 3$

$n \backslash r$	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
2	S	S	F	F	F	F	F	F	T	T	T	W	W	$\cdots$
3	S	S	F	F	F	W	T	T	W	W	W	W	W	$\cdots$
4	S	S	F	F	F	W	W	W	W	W	W	W	W	$\cdots$
5	S	S	F	F	F	W	W	W	W	W	W	W	W	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

- $\overline{S(2,12)}$ and $\overline{S(2,13)}$ are brick infinite ($\Leftarrow \Delta_1$)
- $\overline{S(3,6)}$ is brick infinite ($\Leftarrow \Delta_1$)
- $\overline{S(3,10)}$ and $\overline{S(3,11)}$ are brick infinite ($\Leftarrow \Delta_3$)
- $\overline{S(4,7)}$ is brick infinite ($\Leftarrow \Delta_2$)
- $\overline{S(4,8)}$ is brick infinite ($\Leftarrow \Delta_1$)

char $\mathbb{F} \geq 5$

$n \setminus r$	$1 \sim p-1$	$p \sim 2p-1$	$2p \sim p^2-1$	$p^2 \sim p^2+p-1$	$p^2+p \sim \infty$
2	S	F	F	W	W
3	S	F	W	W	W
4	S	F	W	W	W
5	S	F	W	W	W
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

- $\overline{S(2, p^2+p)}$ and $\overline{S(2, p^2+p+1)}$ are brick infinite ($\Leftarrow \Delta_1$)
- $\overline{S(3, 2p)}$, $\overline{S(3, 2p+1)}$ and $\overline{S(3, 2p+2)}$ are brick infinite ($\Leftarrow \Delta_3$).

All wild cases are solved in [W., 2020], except for

$$(\star)\quad \begin{cases} p = 2, n = 2, r = 8, 17, 19; \\ p = 2, n = 3, r = 4; \\ p = 2, n \geqslant 5, r = 5; \\ p \geqslant 5, n = 2, p^2 \leqslant r \leqslant p^2 + p - 1. \end{cases}$$

and these 4 cases are solved in [Aoki-W., 2021].

References

[AIR] T. Adachi, O. Iyama and I. Reiten, τ-tilting theory. *Compos. Math.* **150** (2014), no. 3, 415–452.

[DIJ] L. Demonet, O. Iyama and G. Jasso, τ-tilting finite algebras, bricks and g-vectors. *Int. Math. Res. Not.* (2017), 1–41.

[E] K. Erdmann, Schur algebras of finite type. *Quart. J. Math. Oxford Ser.* **44** (1993), no. 173, 17–41.

[J] G.D. James, The representation theory of the symmetric groups. Lecture Notes in Mathematics, 682. *Springer, Berlin*, 1978.

[X] C. Xi, On representations types of q-Schur algebras. *J. Pure Appl. Algebra* **84** (1993), no. 1, 73–84.

Thank you for listening!