Representation type of cyclotomic quiver Hecke algebras in affine type A[1]

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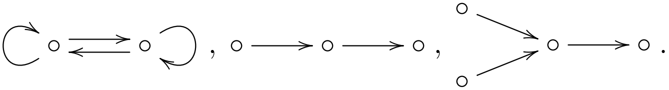
[1] This is joint work with Susumu Ariki and Linliang Song.

Outline

Introduction

We start with Quiver Representation Theory.

Quivers:



A quiver representation:

$$V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3 .$$

We may study quiver rep's algebraically or geometrically.

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- **Algebraic viewpoint**:

- **Geometric viewpoint**:

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- **Algebraic viewpoint**: to find all indecomposable rep's.

  e.g., the above example has 6 indecomposable rep's:

$$K \xrightarrow{0} 0 \xrightarrow{0} 0 \qquad K \xrightarrow{1} K \xrightarrow{0} 0$$
$$0 \xrightarrow{0} K \xrightarrow{0} 0 \qquad 0 \xrightarrow{0} K \xrightarrow{1} K$$
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  where K is a field (algebraically closed).

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  where K is a field (algebraically closed).

- **Geometric viewpoint**: to fix all vector spaces V_i and change matrices f, g. This gives an affine module variety.

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An algebra A is said to be

- **rep-finite** if the number of indecomposable rep's is finite.
- **tame** if it is not rep-finite, but all indecomposable rep's can be organized in a one-parameter family in each dimension.

Otherwise, A is called **wild**.

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Theorem (Drozd 1977)

The representation type of any algebra (over K) is exactly one of rep-finite, tame and wild.

Example: tame algebras

e.g., $\circ \rightrightarrows \circ$ is tame. Indecomposable rep's:

$$\text{dimension 2:} \quad K \underset{0}{\overset{1}{\rightrightarrows}} K \qquad K \underset{\lambda}{\overset{1}{\rightrightarrows}} K$$

$$\text{dimension 3:} \quad K^2 \underset{(1,0)}{\overset{(1,0)}{\rightrightarrows}} K \qquad K \underset{(0,1)^t}{\overset{(1,0)^t}{\rightrightarrows}} K^2$$

$$\text{dimension 4:} \quad K^2 \underset{J_2(0)}{\overset{I_2}{\rightrightarrows}} K^2 \qquad K^2 \underset{J_2(\lambda)}{\overset{I_2}{\rightrightarrows}} K^2$$

$$\vdots$$

$$K^{n+1} \underset{[O,I_n]}{\overset{[I_n,O]}{\rightrightarrows}} K^n \qquad K^n \underset{J_n(\lambda)}{\overset{I_n}{\rightrightarrows}} K^n$$

Example: wild algebras

e.g., $\circ \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; \circ$. Indecomposable rep's:

$$\text{dimension 3:} \qquad K^2 \;\substack{\xrightarrow{(1,0)} \\ \xrightarrow{a} \\ \xrightarrow[(0,1)]{}}\; K \qquad a = (\lambda, \mu)$$

Impossible! to give a complete classification of indecomposable rep's for a wild algebra.

Some examples related to Hecke algebras.

- rep-finite: e.g., Brauer tree algebras
- tame: e.g., Brauer graph algebras
- wild:

KLR algebras in affine type A

Hecke algebras of type A

The symmetric group $\mathfrak{S}_n$ (= permutation group of $\{1, 2, \cdots, n\}$) is generated by $\{s_i = (i, i+1) \mid 1 \le i \le n-1\}$ subject to

$$s_i^2 = 1, (\Leftrightarrow (s_i + 1)(s_i - 1) = 0)$$

$$s_i s_j = s_j s_i \text{ if } |i - j| \neq 1, \quad s_i s_j s_i = s_j s_i s_j \text{ if } |i - j| = 1.$$

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The Iwahori-Hecke algebra $\mathcal{H}(\mathfrak{S}_n)$ is the $\mathbb{Z}[q, q^{-1}]$-algebra generated by $\{T_i \mid 1 \leq i \leq n-1\}$ subject to

$$T_i^2 = (q-1)T_i + q, (\Leftrightarrow (T_i + 1)(T_i - q) = 0)$$

$$T_i T_j = T_j T_i \text{ if } |i - j| \neq 1, \quad T_i T_j T_i = T_j T_i T_j \text{ if } |i - j| = 1.$$

More Hecke algebras

In the last fifty years, the representation theory of symmetric groups had a close connection with Lie theory via categorification.

- Hecke algebras of Coxeter groups, i.e., of type B, D, E, etc.
- Cyclotomic Hecke algebras (a.k.a. Ariki-Koike algebras). See [Ariki-Koike, 1994], [Broue-Malle, 1993], and [Cherednik 1987].
- Cyclotomic quiver Hecke algebras (a.k.a. Cyclotomic KLR algebras). See [Khovanov-Lauda, 2009] and [Rouquier, 2008].

Many classes of algebras arise in this process, whose representation type is completely determined, in particular, for

(1) Hecke alg's in type ABD (Ariki, 2000);

(2) Cyclotomic quiver Hecke alg's of level 1 in affine type ACD (Ariki-Iijima-Park 2014, 2015); of level 2 in affine type A (Ariki 2017);

(3) Schur/q-Schur/Borel-Schur/infinitesimal-Schur alg's (Xi 1993, Erdmann 1993, Doty-Erdmann-Martin 1999, Erdmann-Nakano 2001, etc);

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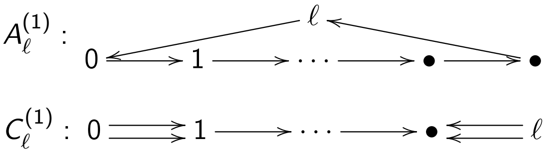
Lie theoretic data

Let $I = \{0, 1, ..., \ell\}$ be an index set. Recall that



$$+B_\ell^{(1)}, D_\ell^{(1)}, A_{2\ell}^{(2)}, A_{2\ell-1}^{(2)}, D_{\ell+1}^{(2)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, F_4^{(1)}, G_2^{(1)}, E_6^{(2)}, D_4^{(3)}.$$

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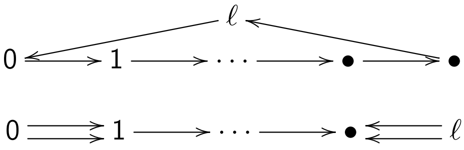

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Set $n_{ij} := \#(i \to j)$.

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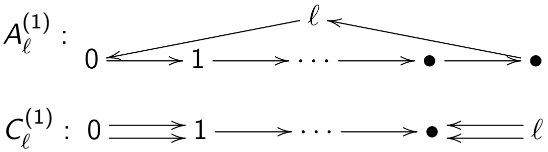

$$+B_\ell^{(1)}, D_\ell^{(1)}, A_{2\ell}^{(2)}, A_{2\ell-1}^{(2)}, D_{\ell+1}^{(2)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, F_4^{(1)}, G_2^{(1)}, E_6^{(2)}, D_4^{(3)}.$$

Set $n_{ij} := \#(i \to j)$. We define the Cartan matrix $A = (a_{ij})_{i,j \in I}$ by

$$a_{ii} = 2, \quad a_{ij} = \begin{cases} -n_{ij} & \text{if } n_{ij} > n_{ji} \\ -1 & \text{if } n_{ij} < n_{ji} \\ -n_{ij} - n_{ji} & \text{otherwise} \end{cases} \quad (i \neq j).$$

Let $(A, P, \Pi, P^\vee, \Pi^\vee)$ be the Cartan datum in type $A_\ell^{(1)}$, where

- $P = \bigoplus_{i=0}^{\ell} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta$ is the weight lattice;
- $\Pi = \{\alpha_i \mid i \in I\} \subset P$ is the set of simple roots;
- $P^\vee = \mathrm{Hom}(P, \mathbb{Z})$ is the coweight lattice;
- $\Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee$ is the set of simple coroots.

The null root is $\delta = \alpha_0 + \alpha_1 + \ldots + \alpha_\ell$. We have

$$\langle h_i, \alpha_j \rangle = a_{ij}, \langle h_i, \Lambda_j \rangle = \delta_{ij} \quad \text{for all } i, j \in I.$$

We set $P^+ := \{\Lambda \in P \mid \langle h_i, \Lambda \rangle \in \mathbb{Z}_{\geq 0}, i \in I\}$.

A family of polynomials in type A

Fix $t \in K$ if $\ell = 1$ and $0 \neq t \in K$ if $\ell \geq 2$.

For $i, j \in I$, we take $Q_{i,j}(u, v) \in K[u, v]$ such that $Q_{i,i}(u, v) = 0$, $Q_{i,j}(u, v) = Q_{j,i}(v, u)$ and if $\ell \geq 2$,

$$\begin{aligned} Q_{i,i+1}(u, v) &= u + v \text{ if } 0 \leq i < \ell, \\ Q_{\ell,0}(u, v) &= u + tv, \\ Q_{i,j}(u, v) &= 1 \text{ if } j \not\equiv_e i, i \pm 1. \end{aligned}$$

If $\ell = 1$, we take $Q_{0,1}(u, v) = u^2 + tuv + v^2$.

Quiver Hecke algebras

The quiver Hecke algebra $R(n)$ associated with $(Q_{i,j}(u,v))_{i,j\in I}$ is the $\mathbb{Z}$-graded $\Bbbk$-algebra generated by

$$\{e(\nu) \mid \nu = (\nu_1, \nu_2, \dots, \nu_n) \in I^n\}, \{x_i \mid 1 \le i \le n\}, \{\psi_j \mid 1 \le j \le n-1\},$$

subject to the following relations:

(1) $e(\nu)e(\nu') = \delta_{\nu,\nu'} e(\nu),\ \sum_{\nu \in I^n} e(\nu) = 1,\ x_i x_j = x_j x_i,\ x_i e(\nu) = e(\nu) x_i.$

(2) $\psi_i e(\nu) = e(s_i(\nu))\psi_i,\ \psi_i\psi_j = \psi_j\psi_i$ if $|i-j| > 1$.

(3) $\psi_i^2 e(\nu) = Q_{\nu_i,\nu_{i+1}}(x_i, x_{i+1})e(\nu)$.

(4) $(\psi_i x_j - x_{s_i(j)}\psi_i)e(\nu) = \begin{cases} -e(\nu) & \text{if } j = i \text{ and } \nu_i = \nu_{i+1}, \\ e(\nu) & \text{if } j = i+1 \text{ and } \nu_i = \nu_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$

(5) $(\psi_{i+1}\psi_i\psi_{i+1} - \psi_i\psi_{i+1}\psi_i)e(\nu) = \begin{cases} \frac{Q_{\nu_i,\nu_{i+1}}(x_i,x_{i+1}) - Q_{\nu_i,\nu_{i+1}}(x_{i+2},x_{i+1})}{x_i - x_{i+2}} e(\nu) & \text{if } \nu_i = \nu_{i+2}, \\ 0 & \text{otherwise.} \end{cases}$

Cyclotomic quiver Hecke algebras

Fix $\Lambda \in P^+$. The cyclotomic quiver Hecke algebra $R^\Lambda(n)$ w.r.t. Λ is defined as the quotient of $R(n)$ modulo the relation

$$x_1^{\langle h_{\nu_1}, \Lambda \rangle} e(\nu) = 0.$$

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Let $Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$. For each $\beta \in Q_+$ with $|\beta| = n$, we define

$$R^\Lambda(\beta) := e(\beta) R^\Lambda(n) e(\beta),$$

where $e(\beta) := \sum_{\nu \in I^\beta} e(\nu)$ with $I^\beta = \left\{ \nu = (\nu_1, \nu_2, \ldots, \nu_n) \in I^n \mid \sum_{i=1}^{n} \alpha_{\nu_i} = \beta \right\}$.

An example

Set $\Lambda = k\Lambda_0, \ell = 2$. Then, $I = \{0, 1, 2\}$ and $R(3)$ is generated by

$$\{e(000), \cdots, e(012), \cdots, e(212), \cdots\}, \{x_1, x_2, x_3\}, \{\psi_1, \psi_2\},$$

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Set $\beta = \alpha_1 + \alpha_2 + \alpha_3$. Then, $R^\Lambda(\beta)$ is generated by

$$\{e(012), e(021), e(102), e(120), e(201), e(210)\}, \{x_1, x_2, x_3\}, \{\psi_1, \psi_2\},$$

subject to

- $e(102) = e(120) = e(201) = e(210) = 0$, $x_1^k e(012) = x_1^k e(021) = 0$;
- $\psi_1 e(012) = \psi_1 e(021) = 0$, $\psi_2 e(012) = e(021)\psi_2$;
- $x_2 e(012) = -x_1 e(012)$, $x_2 e(021) = -t x_1 e(021)$;
- $x_3^2 e(012) = t x_1^2 e(012) + (1 - t) x_1 x_3 e(012)$, etc.

Known results on cyclotomic KLR algebras

We know the representation type of cyclotomic KLR algebras in the following cases.

- $R^{\Lambda_0}(\beta)$ in type $A^{(2)}_{2\ell}$, see [Ariki-Park, 2014].
- $R^{\Lambda_0}(\beta)$ in type $A^{(1)}_{\ell}$, see [Ariki-Iijima-Park, 2015].
- $R^{\Lambda_0}(\beta)$ in type $C^{(1)}_{\ell}$, see [Ariki-Park, 2015].
- $R^{\Lambda_0}(\beta)$ in type $D^{(2)}_{\ell+1}$, see [Ariki-Park, 2016].
- $R^{\Lambda_0+\Lambda_s}(\beta)$ in type $A^{(1)}_{\ell}$, see [Ariki, 2017].

In this talk, we explain the representation type of $R^{\Lambda}(\beta)$ in type $A^{(1)}_{\ell}$, for arbitrary $\Lambda \in P^+$.

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$$s_i^2 = 1, s_i s_j = s_j s_i \text{ if } |i-j| \not\equiv_{\ell+1} 1, s_i s_j s_i = s_j s_i s_j \text{ if } |i-j| \equiv_{\ell+1} 1.$$

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- A weight $\mu \in P(\Lambda)$ is maximal if $\mu + \delta \notin P(\Lambda)$. We define

$$\max{}^+(\Lambda) := \{\mu \in P^+ \mid \mu \text{ is maximal}\}.$$

Kac's result tells us that the representatives of W-orbits in $P(\Lambda)$ are given by $\{\mu - m\delta \mid \mu \in \max^+(\Lambda), m \in \mathbb{Z}_{\geq 0}\}$.

$\max^+(\Lambda)$

We briefly recall the construction in [Kim-Oh-Oh, 2020] as follows.

Set $\Lambda = a_{i_1}\Lambda_{i_1} + a_{i_2}\Lambda_{i_2} + \cdots + a_{i_n}\Lambda_{i_n} \in P^+$. We define

$$\mathrm{le}(\Lambda) = \sum a_{i_j} \quad \text{and} \quad \mathrm{ev}(\Lambda) = i_1 + i_2 + \cdots + i_n.$$

Suppose $\mathrm{le}(\Lambda) = k$. Then,

$$P^+_{cl,k}(\Lambda) = \left\{\Lambda' \in P^+ \mid \mathrm{le}(\Lambda) = \mathrm{le}(\Lambda'), \mathrm{ev}(\Lambda) \equiv_{\ell+1} \mathrm{ev}(\Lambda') \right\}.$$

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e.g., $P^+_{cl,3}(\Lambda_0 + \Lambda_3 + \Lambda_6)$ with $\ell = 6$ consists of $\Lambda_0 + \Lambda_3 + \Lambda_6$, $\Lambda_1 + \Lambda_2 + \Lambda_6$, $\Lambda_1 + \Lambda_3 + \Lambda_5$, $\Lambda_0 + \Lambda_4 + \Lambda_5$, $\Lambda_2 + \Lambda_3 + \Lambda_4$, $2\Lambda_0 + \Lambda_2$, $\Lambda_4 + 2\Lambda_6$, $2\Lambda_5 + \Lambda_6$, $\Lambda_0 + 2\Lambda_1$, $2\Lambda_2 + \Lambda_5$, $\Lambda_1 + 2\Lambda_4$, $2\Lambda_0 + \Lambda_2$, $3\Lambda_3$.

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Recall that $\langle h_i, \Lambda_j \rangle = \delta_{ij}$. We define $y_i := \langle h_i, \Lambda - \Lambda' \rangle$ and

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Then, we consider the linear equation $AX^t = Y^t_{\Lambda'}$.

Proposition (Ariki-Song-W. 2023)

- The linear equation $AX^t = Y^t_{\Lambda'}$ has a unique solution $X = (x_0, \ldots, x_\ell)$ satisfying

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- The inverse map $\phi_\Lambda^{-1} : P^+_{cl,k}(\Lambda) \to \max^+(\Lambda)$ of ϕ_Λ is given by

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where X is the unique solution of $AX^t = Y^t_{\Lambda'}$ as above.

Set $\beta_{\Lambda'} := \sum_{i \in I} x_i \alpha_i$. Then,

$$\max{}^+(\Lambda) = \left\{ \Lambda - \beta_{\Lambda'} \mid \Lambda' \in P^+_{cl,k}(\Lambda) \right\}.$$

Strategy to prove the results

If $\Lambda - \beta$ lies in the W-orbit of $P(\Lambda)$, then

$$\Lambda - \beta \in \{\Lambda - \beta_{\Lambda'} - m\delta \mid \Lambda' \in P^{+}_{cl,k}(\Lambda), m \in \mathbb{Z}_{\geq 0}\}.$$

Thus, we only need to consider $R^{\Lambda}(\beta)$ for $\beta = \beta_{\Lambda'} + m\delta$ with $\Lambda' \in P^{+}_{cl,k}(\Lambda)$ and $m \in \mathbb{Z}_{\geq 0}$.

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Thus, we only need to consider $R^{\Lambda}(\beta)$ for $\beta = \beta_{\Lambda'} + m\delta$ with $\Lambda' \in P^{+}_{cl,k}(\Lambda)$ and $m \in \mathbb{Z}_{\geq 0}$.

Step 1: We show that $R^{\Lambda}(\beta_{\Lambda'} + m\delta)$ is wild for all $m \geq 1$ if $\beta_{\Lambda'} \neq 0$ and $R^{\Lambda}(m\delta)$ is wild for all $m \geq 2$, by using some new reduction theorems.

(If $R^{\Lambda}(\gamma)$ is not wild, we set $\gamma \in \mathcal{NW}(\Lambda) \cup \{\delta\}$.)

Step 2: We determine the representation type of $R^\Lambda(\gamma)$ for $\gamma \in \mathcal{T}(\Lambda) \cup \{\delta\}$, via case-by-case consideration.

(A systematic approach developed by Ariki and his collaborators is well applied to find the quiver presentation of $R^\Lambda(\gamma)$.)

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Step 3: We show that

$$\mathcal{NW}(\Lambda) \subset \mathcal{T}(\Lambda)$$

via case-by-case consideration on small k (i.e., $k = 3, 4, 5, 6$) and via induction on $k \geq 7$.

Structure of $P^{+}_{cl,k}(\Lambda)$

(in type $A^{(1)}_{\ell}$)

Recall that

$$\max^+(\Lambda) = \left\{ \Lambda - \beta_{\Lambda'} \mid \Lambda' \in P^+_{cl,k}(\Lambda) \right\}.$$

e.g., $P^+_{cl,3}(\Lambda_0 + \Lambda_3 + \Lambda_6)$ with $\ell = 6$ consists of $\Lambda_0 + \Lambda_3 + \Lambda_6$, $\Lambda_1 + \Lambda_2 + \Lambda_6$, $\Lambda_1 + \Lambda_3 + \Lambda_5$, $\Lambda_0 + \Lambda_4 + \Lambda_5$, $\Lambda_2 + \Lambda_3 + \Lambda_4$, etc.

For any $\Lambda' \in P^+_{cl,k}(\Lambda)$ with $k \geq 2$, we can write $\Lambda' = \Lambda_i + \Lambda_j + \tilde{\Lambda}$ for some $i, j \in I$ and $\tilde{\Lambda} \in P^+_{cl,k-2}$. Then, we define

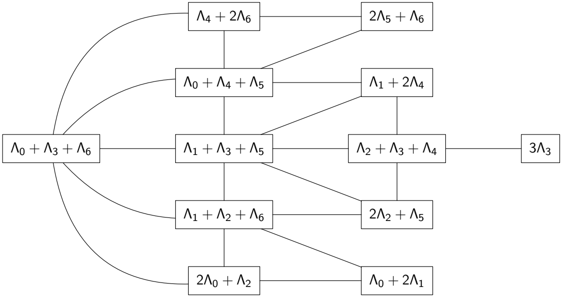
$$\Lambda'_{i,j} := \Lambda_{i-1} + \Lambda_{j+1} + \tilde{\Lambda}.$$

Note that $\Lambda'_{i,j} = \Lambda'$ if and only if $j \equiv_e i - 1$.

Definition 3.1

Let $C(\Lambda)$ be an undirected graph, where we draw an edge between Λ' and Λ'' if $\Lambda'' = \Lambda'_{i,j}$ for some $i, j \in I$ with $j \not\equiv_e i - 1$.

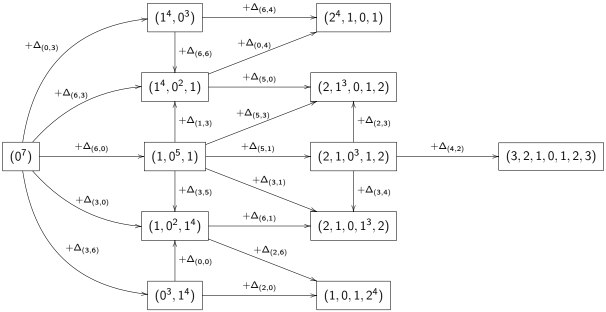
e.g., $C(\Lambda_0 + \Lambda_3 + \Lambda_6)$ with $\ell = 6$ is displayed as follows.



We define

$$\Delta_{i,j} = \begin{cases} (0^i, 1^{j-i+1}, 0^{\ell-j}) & \text{if } i \leq j, \\ (1^{j+1}, 0^{i-j-1}, 1^{\ell-i+1}) & \text{if } i > j. \end{cases}$$

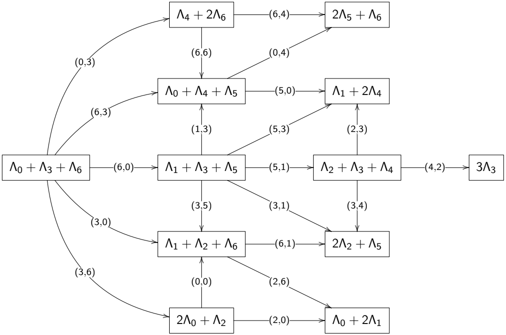
The unique solution of $AX^t = Y^t_{\Lambda'}$ is given by $\min(X_{\Lambda'} + \Delta_{i,j}) = 0$. e.g.,



Definition 3.2

Let $\vec{C}(\Lambda)$ be the quiver where we set $\Lambda' \to \Lambda''$ if $X_{\Lambda''} = X_{\Lambda'} + \Delta_{i,j}$. We label this arrow by (i, j).

e.g., $\vec{C}(\Lambda_0 + \Lambda_3 + \Lambda_6)$ with $\ell = 6$ is displayed as



Proposition 3.3

For any $\Lambda' \in P^+_{cl,k}(\Lambda)$ with $\Lambda' \neq \Lambda$, there is a directed path from Λ to Λ' in $\vec{C}(\Lambda)$. In particular, $\vec{C}(\Lambda)$ is a finite-connected quiver.

Proposition 3.3

For any $\Lambda' \in P^+_{cl,k}(\Lambda)$ with $\Lambda' \neq \Lambda$, there is a directed path from Λ to Λ' in $\vec{C}(\Lambda)$. In particular, $\vec{C}(\Lambda)$ is a finite-connected quiver.

Proposition 3.4

Suppose $\Lambda = \bar{\Lambda} + \tilde{\Lambda}$. Then, there is a directed path

$$\Lambda^{(1)} \xrightarrow{(i_1,j_1)} \Lambda^{(2)} \xrightarrow{(i_2,j_2)} \dots \xrightarrow{(i_{m-1},j_{m-1})} \Lambda^{(m)} \in \vec{C}(\bar{\Lambda})$$

if and only if there is a directed path

$$\Lambda^{(1)} + \tilde{\Lambda} \xrightarrow{(i_1,j_1)} \Lambda^{(2)} + \tilde{\Lambda} \xrightarrow{(i_2,j_2)} \dots \xrightarrow{(i_{m-1},j_{m-1})} \Lambda^{(m)} + \tilde{\Lambda} \in \vec{C}(\Lambda).$$

Key Lemmas

Lemma 3.5

Suppose that there is an arrow $\Lambda' \xrightarrow{(i,j)} \Lambda''$ in $\vec{C}(\Lambda)$. If $R^{\Lambda}(\beta_{\Lambda'})$ is representation-infinite (resp. wild), then so is $R^{\Lambda}(\beta_{\Lambda''})$.

Key Lemmas

Lemma 3.5

Suppose that there is an arrow $\Lambda' \xrightarrow{(i,j)} \Lambda''$ in $\vec{C}(\Lambda)$. If $R^{\Lambda}(\beta_{\Lambda'})$ is representation-infinite (resp. wild), then so is $R^{\Lambda}(\beta_{\Lambda''})$.

Lemma 3.6

Write $\Lambda = \bar{\Lambda} + \tilde{\Lambda}$. If $R^{\bar{\Lambda}}(\beta)$ is representation-infinite (resp. wild), then $R^{\Lambda}(\beta)$ is representation-infinite (resp. wild).

Rep-finite and tame sets

Set $i_0 := i_h$, $i_{h+1} := i_1$ and write

$$\Lambda = m_{i_1}\Lambda_{i_1} + \cdots + m_{i_j}\Lambda_{i_j} + m_{i_{j+1}}\Lambda_{i_{j+1}} + \cdots + m_{i_h}\Lambda_{i_h}$$

Rep-finite and tame sets

Set $i_0 := i_h$, $i_{h+1} := i_1$ and write

$$\Lambda = m_{i_1}\Lambda_{i_1} + \cdots + m_{i_j}\Lambda_{i_j} + m_{i_{j+1}}\Lambda_{i_{j+1}} + \cdots + m_{i_h}\Lambda_{i_h}$$

For any $1 \leq j \leq h$, we define

$$\begin{aligned}
F(\Lambda)_0 &:= \{\Lambda_{i_j,i_j} \mid m_{i_j} = 2\} \\
F(\Lambda)_1 &:= \{\Lambda_{i_j,i_{j+1}} \mid m_{i_j} = 1, m_{i_{j+1}} = 1\} \\
T(\Lambda)_1 &:= \{\Lambda_{i_j,i_{j+1}} \mid m_{i_j} = 1, m_{i_{j+1}} > 1 \text{ or } m_{i_j} > 1, m_{i_{j+1}} = 1\} \\
T(\Lambda)_2 &:= \{(\Lambda_{i_j,i_j})_{i_j-1,i_j+1} \mid m_{i_j} = 2, i_{j-1} \not\equiv_e i_j - 1, i_{j+1} \not\equiv_e i_j + 1\} \text{ if char } K \neq 2 \\
T(\Lambda)_3 &:= \{(\Lambda_{i_j,i_j})_{i_j,i_j+1 \text{ or } i_j-1,i_j} \mid m_{i_j} = 3, i_{j+1} \not\equiv_e i_j + 1 \text{ or } i_{j-1} \not\equiv_e i_j - 1\} \text{ if char } K \neq 3 \\
T(\Lambda)_4 &:= \{(\Lambda_{i_j,i_j})_{i_j,i_j} \mid m_{i_j} = 4\} \text{ if char } K \neq 2 \\
T(\Lambda)_5 &:= \{(\Lambda_{i_j,i_j})_{i_p,i_p} \mid m_{i_j} = m_{i_p} = 2, i_p \not\equiv_e i_j \pm 1, j \neq p\}
\end{aligned}$$

Set

$$\mathcal{F}(\Lambda) = \{\beta_{\Lambda'} \mid \Lambda' \in \{\Lambda\} \cup F(\Lambda)_0 \cup F(\Lambda)_1\},$$
$$\mathcal{T}(\Lambda) = \{\beta_{\Lambda'} \mid \Lambda' \in \cup_{1 \leq j \leq 5} T(\Lambda)_j\}.$$

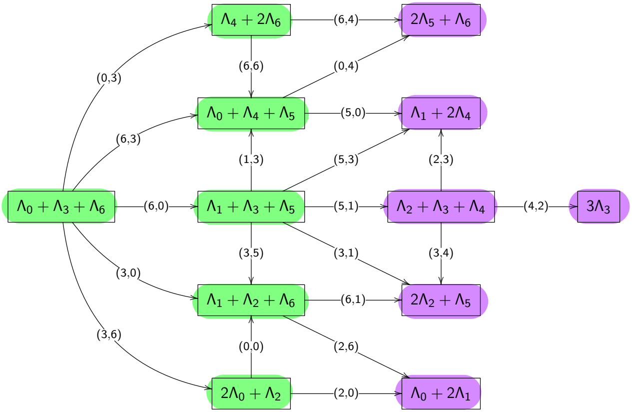
Theorem 3.7 (Ariki-Song-W. 2023)

Suppose $\mathrm{le}(\Lambda) \geq 3$. Then, $R^\Lambda(\beta)$ is representation-finite if $\beta \in \mathcal{F}(\Lambda)$, tame if one of the following holds:

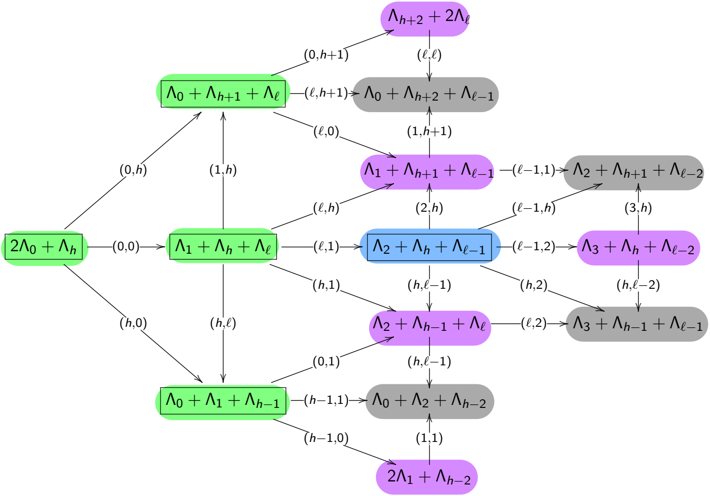
- $\beta = \delta$, $\Lambda = k\Lambda_i$, $\ell = 1$ with $t \neq \pm 2$,
- $\beta = \delta$, $\Lambda = k\Lambda_i$, $\ell \geq 2$ with $t \neq (-1)^{\ell+1}$,
- $\beta \in \mathcal{T}(\Lambda)$.

Otherwise, it is wild.

e.g., rep-type of $\vec{C}(\Lambda_0 + \Lambda_3 + \Lambda_6)$ with $\ell = 6$ is displayed as



e.g., rep-type of $\vec{C}(2\Lambda_0 + \Lambda_h)$ is displayed as



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Thank you! Any questions?

Tools:
- Bound quiver algebras;
- Representation of quivers;
- Representation type: rep-finite, tame, wild;
- Brauer tree/graph algebras.

Objects:
- Symmetric groups and Hecke algebras;
- Lie theoretic data and Cartan datum;
- Quiver Hecke algebras;
- Cyclotomic KLR algebras;
- $\max^+(\Lambda)$ and $P^+_{cl,k}(\Lambda)$;
- Rep-finite and tame sets.