On τ-tilting finiteness of symmetric algebras

Qi WANG
Yau Mathematical Sciences Center
Tsinghua University

ALTReT2023, May 14, 2023

Outline

Introduction

Symmetric algebras

Symmetric algebra: $A \cong D(A)$ as A-A-bimodule.

- K: an algebraically closed field
- $D(-) = \mathrm{Hom}_K(-, K) : \mathrm{mod}\, A \longrightarrow \mathrm{mod}\, A^{\mathrm{op}}$

Some examples:

(1) Group algebras of finite groups;

(2) Brauer tree/graph/configuration algebras;

(3) Hecke algebras associated with Coxeter groups;

(4) Trivial extension $T(A) = A \ltimes D(A)$ of an algebra A.

Representation type of algebras

Theorem 1.1 (Drozd, 1977)

The representation type of any algebra (over K) is exactly one of rep-finite, tame and wild.

An algebra A is said to be

- **rep-finite** if the set of indecomposable modules is finite.
- **tame** if it is not finite, but all indecomposable modules are organized in a one-parameter family in each dimension.

Otherwise, A is called wild.

Some examples of symmetric algebras:

- rep-finite: e.g., Brauer tree algebras
- tame: e.g., Brauer graph algebras
- wild: e.g., trivial extensions of wild hereditary algebras

Impossible! to give a complete classification of indecomposable modules for wild algebras.

τ-tilting finiteness

Part of motivations

To capture **some finite property** in wild cases.

A **brick** M of A provided $\mathrm{End}_A(M) \simeq K$.

τ-tilting finiteness

Part of motivations

To capture **some finite property** in wild cases.

A **brick** M of A provided $\mathrm{End}_A(M) \simeq K$. Then, A is called

(1) (Chindris-Kinser-Weyman, 2012) **Schur-representation-finite** if there are finitely many bricks of a fixed dimension.

(2) (Demonet-Iyama-Jasso, 2016) τ**-tilting finite** if there are finitely many bricks.

τ-tilting finiteness

Part of motivations

To capture **some finite property** in wild cases.

A **brick** M of A provided $\mathrm{End}_A(M) \simeq K$. Then, A is called

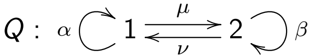
(1) (Chindris-Kinser-Weyman, 2012) **Schur-representation-finite** if there are finitely many bricks of a fixed dimension.

(2) (Demonet-Iyama-Jasso, 2016) **τ-tilting finite** if there are finitely many bricks.

- (2) $\Rightarrow$ (1) is obvious.
- (1) $\Rightarrow$ (2) is not verified; no counterexample.

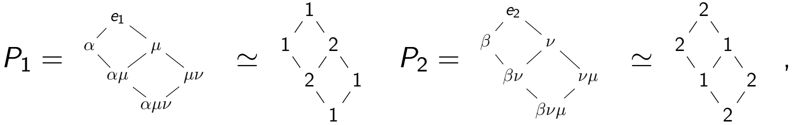
Wild, but τ-tilting finite

e.g., set $\Lambda = KQ/I$ with



$$I = \langle \alpha^2, \beta^2, \mu\nu\mu, \nu\mu\nu, \alpha\mu - \mu\beta, \beta\nu - \nu\alpha \rangle$$

The indecomposable projective Λ-modules are



then Λ is a wild symmetric algebra. But, it has only 4 bricks.

Derived equivalence

- A is Morita equivalent to $B \Leftrightarrow \operatorname{mod} A \simeq \operatorname{mod} B$
- A is Derived equivalent to $B \Leftrightarrow \mathrm{D}^{\mathrm{b}}(\operatorname{mod} A) \simeq \mathrm{D}^{\mathrm{b}}(\operatorname{mod} B)$

Derived equivalence

- A is Morita equivalent to $B \Leftrightarrow \operatorname{mod} A \simeq \operatorname{mod} B$
- A is Derived equivalent to $B \Leftrightarrow \mathrm{D}^{\mathrm{b}}(\operatorname{mod} A) \simeq \mathrm{D}^{\mathrm{b}}(\operatorname{mod} B)$

Theorem 1.2 (Rickard, 1991)

If A is symmetric and $A \simeq_d B$, then B is also symmetric.

Derived equivalence

- A is Morita equivalent to $B \Leftrightarrow \text{mod}\, A \simeq \text{mod}\, B$
- A is Derived equivalent to $B \Leftrightarrow \mathrm{D}^{\mathrm{b}}(\text{mod}\, A) \simeq \mathrm{D}^{\mathrm{b}}(\text{mod}\, B)$

Theorem 1.2 (Rickard, 1991)

If A is symmetric and $A \simeq_d B$, then B is also symmetric.

Theorem 1.3 (Krause, 1997)

If A is symmetric with property $(*) \in \{\text{rep-finite, tame, wild}\}$ and $A \simeq_d B$, then B also admits $(*)$.

Derived equivalence

- A is Morita equivalent to $B \Leftrightarrow \operatorname{mod} A \simeq \operatorname{mod} B$
- A is Derived equivalent to $B \Leftrightarrow \mathrm{D}^{\mathrm{b}}(\operatorname{mod} A) \simeq \mathrm{D}^{\mathrm{b}}(\operatorname{mod} B)$

Theorem 1.2 (Rickard, 1991)

If A is symmetric and $A \simeq_d B$, then B is also symmetric.

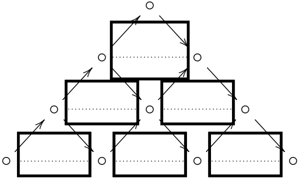
Theorem 1.3 (Krause, 1997)

If A is symmetric with property $(*) \in \{\text{rep-finite, tame, wild}\}$ and $A \simeq_d B$, then B also admits $(*)$.

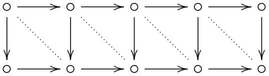
Question: Let $A \simeq_d B$ be two symmetric algebras. Then,

$$A \text{ is } \tau\text{-tilting finite} \Leftrightarrow B \text{ is } \tau\text{-tilting finite ?}$$

In general, derived equivalence will not preserve τ-tilting finiteness. e.g.,



$\simeq_d$



τ-tilting finite $\qquad$ τ-tilting infinite

τ-tilting theory

τ-tilting theory was introduced by Adachi, Iyama and Reiten in 2014, as a completion to the classical tilting theory.

So far, τ-tilting theory is related to several different aspects in Representation Theory of Algebras:

- Categorical objects, such as torsion classes, silting complexes;
- Combinatorial objects, such as bricks, semibricks;
- Lattice theory, such as the lattice of torsion classes;
- Geometric objects, such as the modern Brauer-Thrall conjecture.

Auslander-Reiten translation

- $(-)^* = \mathrm{Hom}_A(-, A) : \mathrm{proj}\ A \longleftrightarrow \mathrm{proj}\ A^{\mathrm{op}}$
- the Nakayama functor $\nu = D(-)^* : \mathrm{proj}\ A \to \mathrm{inj}\ A$

For an A-module M with a minimal projective presentation

$$P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0,$$

the **Auslander-Reiten translation** τM is defined by the following exact sequence

$$0 \longrightarrow \tau M \longrightarrow \nu P_1 \xrightarrow{\nu d_1} \nu P_0.$$

- $|M|$: the number of isomorphism classes of indecomposable direct summands of M

Definition 1.1 (Adachi-Iyama-Reiten, 2014)

Let M be a right A-module. Then,

(1) M is called τ-rigid if $\mathrm{Hom}_A(M, \tau M) = 0$.

(2) M is called τ-tilting if M is τ-rigid and $|M| = |A|$.

(3) M is called support τ-tilting if there exists an idempotent e of A such that M is a τ-tilting (A/AeA)-module.

An algebra A is called **τ-tilting finite** if the set $\mathrm{s}\tau\text{-tilt}\, A$ is finite.

Mutation

Reminder: $T_1 \oplus \ldots \oplus T_j \oplus \ldots \oplus T_n \Rightarrow T_1 \oplus \ldots \oplus T_j^* \oplus \ldots \oplus T_n$.

For $M, N \in \mathsf{s}\tau\text{-tilt}\, A$, we say $M \geqslant N$ if $\mathrm{Fac}(N) \subseteq \mathrm{Fac}(M)$.

- $\mathrm{Fac}(M)$: the full subcategory whose objects are factor modules of finite direct sums of copies of M

Definition 1.2 (Adachi-Iyama-Reiten, 2014)

For $M, N \in \mathsf{s}\tau\text{-tilt}\, A$, the following conditions are equivalent.

- N is a left mutation of M.
- M is a right mutation of N.
- $M > N$ and there is no $T \in \mathsf{s}\tau\text{-tilt}\, A$ such that $M > T > N$.

Connection with bricks

- brick A: the set of (iso. classes of) bricks in $\mathrm{mod}\, A$
- fbrick A: the set of bricks M satisfying the condition that the smallest torsion class $\mathsf{T}(M)$ containing M is functorially finite.

Theorem 1.3 (Demonet-Iyama-Jasso, 2016)

There exists a bijection between $i\tau$-rigid A and fbrick A given by

$$X \mapsto X/\mathrm{rad}_B(X),$$

where $B := \mathrm{End}_A(X)$. In τ-tilting finite case, brick A = fbrick A.

Remark: each indecomposable τ-rigid module is a direct summand of some (support) τ-tilting modules.

Reduction theorems

Proposition 1.4 (Demonet-Iyama-Jasso, 2016)

If A is τ-tilting finite, then

(1) A/I is τ-tilting finite for any two-sided ideal I of A.

(2) eAe is τ-tilting finite for any idempotent e of A.

Reduction theorems

Proposition 1.4 (Demonet-Iyama-Jasso, 2016)

If A is τ-tilting finite, then

(1) A/I is τ-tilting finite for any two-sided ideal I of A.

(2) eAe is τ-tilting finite for any idempotent e of A.

Recall that $T(A) = A \ltimes D(A)$ is the trivial extension of A.

Corollary 1.5

If A is τ-tilting infinite, then so is $T(A)$.

Brauer graph algebras

Let A be a Brauer graph algebra with Brauer graph Γ_A.

- Γ_A is called bipartite if every cycle in Γ_A has an even length

Brauer graph algebras

Let A be a Brauer graph algebra with Brauer graph Γ_A.

- Γ_A is called bipartite if every cycle in Γ_A has an even length

Theorem 1.6 (Adachi-Aihara-Chan, 2018)

A is τ-tilting finite if and only if Γ_A contains at most one odd cycle and no even cycle.

Brauer graph algebras

Let A be a Brauer graph algebra with Brauer graph Γ_A.

- Γ_A is called bipartite if every cycle in Γ_A has an even length

Theorem 1.6 (Adachi-Aihara-Chan, 2018)

A is τ-tilting finite if and only if Γ_A contains at most one odd cycle and no even cycle.

Theorem 1.7 (Opper-Zvonareva, 2022)

$A \simeq_d B$ if and only if the following conditions hold.

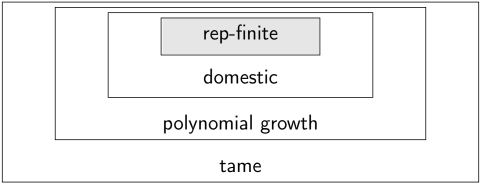
(1) Γ_A and Γ_B share the same number of vertices, edges, faces,

(2) the multisets of multiplicities and the multisets of perimeters of faces of Γ_A and Γ_B coincide,

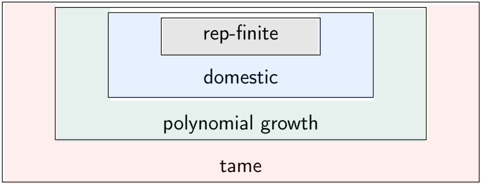
(3) either both or none of Γ_A and Γ_B are bipartite.

Symmetric algebras

Tame algebras have the following hierarchy:



Tame algebras have the following hierarchy:



Theorem 3.1 (Krause-Zwar, 2000)

If A is symmetric with $(*) \in \{\mathrm{D}, \mathrm{PG}\}$ and $A \simeq_d B$, then B also admits $(*)$.

Symmetric algebras of polynomial growth

	S	Cartan matrix	Morita equivalence	Derived equivalence	Class
D	○	singular	trivial extensions of Euclidean algebras	$T(p,q), T(2,2,r)^*$, $T(3,3,3), T(2,4,4)$, $T(2,3,6)$	(1)
	○	non-singular	$\Lambda(T,v_1,v_2), \Lambda'(T)$, $\Gamma^{(0)}(T,v), \Gamma^{(1)}(T,v)$, $\Gamma^{(2)}(T,v_1,v_2)$	$A(p,q), \Lambda(m), \Gamma(n)$	(2)
	×	non-singular	$\Omega(T)$	$\Omega(n)$	(3)
PG	○	singular	trivial extensions of tubular algebras	trivial extensions of canonical tubular algebras	(4)
	○	non-singular	$\{A_i \mid i = 1, 2, \ldots, 16\}$	$\{A_i \mid i = 1, 2, 3, 4, 5, 12\}$	(5)
	×	non-singular	$\{\Lambda_i \mid i = 1, 2, \ldots, 9\}$	$\{\Lambda_i \mid i = 1, 3, 4, 9\}$	(6)

- Standard: A admits a Galois covering with some conditions;
- Cartan matrix: $C_A = (c_{ij})$ with $c_{ij} := \dim \mathrm{Hom}_A(P_i, P_j)$.

Symmetric algebras of polynomial growth

	S	Cartan matrix	Morita equivalence	Derived equivalence	Class
D	○	singular	trivial extensions of Euclidean algebras	$T(p,q)$, $T(2,2,r)^*$, $T(3,3,3)$, $T(2,4,4)$, $T(2,3,6)$	(1)
		non-singular	$\Lambda(T, v_1, v_2), \Lambda'(T)$, $\Gamma^{(0)}(T,v), \Gamma^{(1)}(T,v)$, $\Gamma^{(2)}(T, v_1, v_2)$	$A(p,q), \Lambda(m), \Gamma(n)$	(2)
	×	non-singular	$\Omega(T)$	$\Omega(n)$	(3)
PG	○	singular	trivial extensions of tubular algebras	trivial extensions of canonical tubular algebras	(4)
		non-singular	$\{A_i \mid i = 1,2,\ldots,16\}$	$\{A_i \mid i = 1,2,3,4,5,12\}$	(5)
	×	non-singular	$\{\Lambda_i \mid i = 1,2,\ldots,9\}$	$\{\Lambda_i \mid i = 1,3,4,9\}$	(6)

- Standard: A admits a Galois covering with some conditions;
- Cartan matrix: $C_A = (c_{ij})$ with $c_{ij} := \dim \operatorname{Hom}_A(P_i, P_j)$.

Class (1): τ-tilting infinite

A Euclidean algebra is defined to be a representation-infinite tilted algebra of extended Dynkin types.

Lemma 3.2 (Zito, 2020)

A tilted algebra is τ-tilting finite if and only if it is rep-finite.

Proposition 3.3

All algebras in Class (1) are τ-tilting infinite.

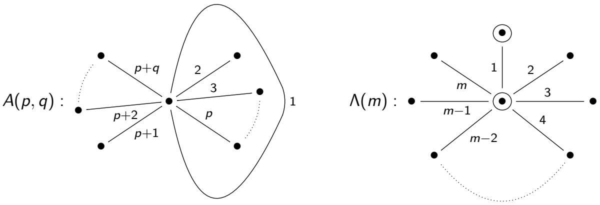
Symmetric algebras of polynomial growth

	S	Cartan matrix	Morita equivalence	Derived equivalence	Class
D	◯	singular	trivial extensions of Euclidean algebras	$T(p,q)$, $T(2,2,r)^*$, $T(3,3,3)$, $T(2,4,4)$, $T(2,3,6)$	(1)
		non-singular	$\Lambda(T,v_1,v_2), \Lambda'(T)$, $\Gamma^{(0)}(T,v), \Gamma^{(1)}(T,v)$, $\Gamma^{(2)}(T,v_1,v_2)$	$A(p,q), \Lambda(m), \Gamma(n)$	(2)
	×	non-singular	$\Omega(T)$	$\Omega(n)$	(3)
PG	◯	singular	trivial extensions of tubular algebras	trivial extensions of canonical tubular algebras	(4)
		non-singular	$\{A_i \mid i = 1, 2, \ldots, 16\}$	$\{A_i \mid i = 1, 2, 3, 4, 5, 12\}$	(5)
	×	non-singular	$\{\Lambda_i \mid i = 1, 2, \ldots, 9\}$	$\{\Lambda_i \mid i = 1, 3, 4, 9\}$	(6)

- Standard: A admits a Galois covering with some conditions;
- Cartan matrix: $C_A = (c_{ij})$ with $c_{ij} := \dim \mathrm{Hom}_A(P_i, P_j)$.

Classes (2) and (3): τ-tilting finite

$A(p,q), \Lambda(m), \Gamma(n), \Omega(n)$: Brauer graph algebras. e.g.,



Remark

$A(p,q), \Lambda(m), \Gamma(n), \Omega(n)$ are τ-tilting finite.

Proposition 3.4

All algebras in Classes (2) and (3) are τ-tilting finite.

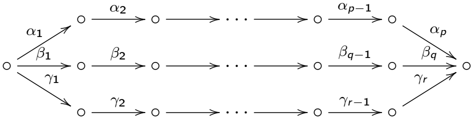
Symmetric algebras of polynomial growth

	S	Cartan matrix	Morita equivalence	Derived equivalence	Class
D	◯	singular	trivial extensions of Euclidean algebras	$T(p,q), T(2,2,r)^*, T(3,3,3), T(2,4,4), T(2,3,6)$	(1)
		non-singular	$\Lambda(T,v_1,v_2), \Lambda'(T), \Gamma^{(0)}(T,v), \Gamma^{(1)}(T,v), \Gamma^{(2)}(T,v_1,v_2)$	$A(p,q), \Lambda(m), \Gamma(n)$	(2)
	×	non-singular	$\Omega(T)$	$\Omega(n)$	(3)
PG	◯	singular	trivial extensions of tubular algebras	trivial extensions of canonical tubular algebras	(4)
		non-singular	$\{A_i \mid i = 1, 2, \dots, 16\}$	$\{A_i \mid i = 1,2,3,4,5,12\}$	(5)
	×	non-singular	$\{\Lambda_i \mid i = 1, 2, \dots, 9\}$	$\{\Lambda_i \mid i = 1,3,4,9\}$	(6)

- Standard: A admits a Galois covering with some conditions;
- Cartan matrix: $C_A = (c_{ij})$ with $c_{ij} := \dim \operatorname{Hom}_A(P_i, P_j)$.

Class (4): τ-tilting infinite

Tubular algebras are representation-infinite simply connected algebras. e.g., the tubular algebra $\mathcal{C}(p,q,r)$ with $p \leqslant q \leqslant r$ is defined by the quiver



bounded by $\alpha_1\alpha_2\cdots\alpha_p + \beta_1\beta_2\cdots\beta_q + \gamma_1\gamma_2\cdots\gamma_r = 0$.

Lemma 3.5 (W. 2019)

A simply connected algebra is τ-tilting finite $\Leftrightarrow$ it is rep-finite.

Proposition 3.6

All algebras in Class (4) are τ-tilting infinite.

Symmetric algebras of polynomial growth

	S	Cartan matrix	Morita equivalence	Derived equivalence	Class
D	○	singular	trivial extensions of Euclidean algebras	$T(p,q)$, $T(2,2,r)^*$, $T(3,3,3)$, $T(2,4,4)$, $T(2,3,6)$	(1)
		non-singular	$\Lambda(T,v_1,v_2), \Lambda'(T)$, $\Gamma^{(0)}(T,v), \Gamma^{(1)}(T,v)$, $\Gamma^{(2)}(T,v_1,v_2)$	$A(p,q), \Lambda(m), \Gamma(n)$	(2)
	×	non-singular	$\Omega(T)$	$\Omega(n)$	(3)
PG	○	singular	trivial extensions of tubular algebras	trivial extensions of canonical tubular algebras	(4)
		non-singular	$\{A_i \mid i = 1, 2, \ldots, 16\}$	$\{A_i \mid i = 1, 2, 3, 4, 5, 12\}$	(5)
	×	non-singular	$\{\Lambda_i \mid i = 1, 2, \ldots, 9\}$	$\{\Lambda_i \mid i = 1, 3, 4, 9\}$	(6)

- Standard: A admits a Galois covering with some conditions;
- Cartan matrix: $C_A = (c_{ij})$ with $c_{ij} := \dim \operatorname{Hom}_A(P_i, P_j)$.

Classes (5) and (6): τ-tilting finite

We check the τ-tilting finiteness of A_i and Λ_i by direct calculation.

$A_1(\lambda)$	$A_2(\lambda)$	A_3	A_4	A_5	A_6	A_7	A_8
24	6	192	132	8	8	108	100

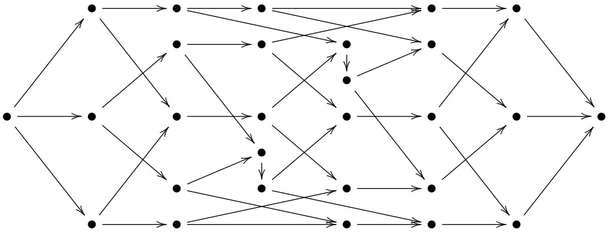
A_9	A_{10}	A_{11}	A_{12}	A_{13}	A_{14}	A_{15}	A_{16}
108	116	100	32	28	32	30	30

Λ_1	Λ_2	$\Lambda_3(\lambda)$	Λ_4	Λ_5	Λ_6	Λ_7	Λ_8	Λ_9
8	8	6	32	28	32	30	30	192

Lemma 3.7 (Adachi-Iyama-Reiten, 2014)

If the Hasse quiver $\mathcal{H}(\mathsf{s}\tau\text{-tilt}\,A)$ contains a finite connected component Δ, then $\mathcal{H}(\mathsf{s}\tau\text{-tilt}\,A) = \Delta$.

e.g., the Hasse quiver of sτ-tilt A_{13}:



Proposition 3.8

All algebras in Classes (5) and (6) are τ-tilting finite.

Main Result

Theorem 3.9 (Aihara-Honma-Miyamoto-W. 2020, Miyamoto-W. 2022)

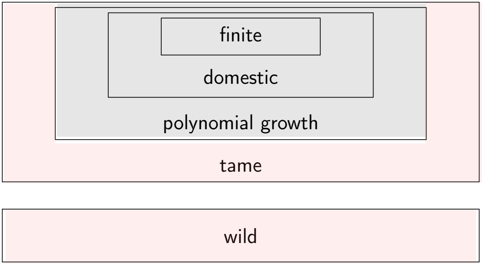
Let $A \simeq_d B$ be two symmetric algebras of polynomial growth.
Then, A is τ-tilting finite if and only if B is τ-tilting finite.

Main Result

Theorem 3.9 (Aihara-Honma-Miyamoto-W. 2020, Miyamoto-W. 2022)

Let $A \simeq_d B$ be two symmetric algebras of polynomial growth. Then, A is τ-tilting finite if and only if B is τ-tilting finite.

Still open:



References

[AAC] T. Adachi, T. Aihara and A. Chan, Classification of two-term tilting complexes over Brauer graph algebras, *Math. Z.*, **290** (2018), no. 2, 1–36.

[AIR] T. Adachi, O. Iyama and I. Reiten, τ-tilting theory. *Compos. Math.* **150** (2014), no. 3, 415–452.

[DIJ] L. Demonet, O. Iyama and G. Jasso, τ-tilting finite algebras, bricks and g-vectors. *Int. Math. Res. Not.* (2017), pp. 1–41.

[S] A. Skowroński, Selfinjective algebras: finite and tame type, Trends in Representation Theory of Algebras and Related Topics, Contemp. Math. **406**, *Amer. Math. Soc.*, 2006.

Thank you! Any questions?

Tools
- Representation type: finite, tame, wild;
- Morita and Derived equivalences;
- τ-tilting finiteness.

Objects
- Symmetric algebras;
- Brauer graph algebras;
- Titled and simply connected algebras.

How?
- Quotient and idempotent truncation;
- Trivial extensions;
- Mutation and mutation graph.