A symmetry of two-term silting quivers[1]

Qi Wang

Yau Mathematical Sciences Center
Tsinghua University

Algebraic Lie Theory and Representation Theory 2022

May 25, 2022

[1] This is joint work with Takuma Aihara, see arXiv: 2205.00472.

Outline

Introduction

We work with a finite-dimensional algebra Λ over an algebraically closed field K. Tilting theory plays a fundamental role in the representation theory of Λ, especially, in constructing derived equivalence classes of Λ.

Theorem (Rickard, 1989)

Let $\mathrm{D^b}(\mathrm{mod}\ \Lambda)$ be the bounded derived category of mod Λ. Then, Γ is derived equivalent to Λ, i.e.,

$$\mathrm{D^b}(\mathrm{mod}\ \Lambda) \xrightarrow{\sim} \mathrm{D^b}(\mathrm{mod}\ \Gamma),$$

if and only if there is a tilting complex T in $\mathrm{K^b}(\mathrm{proj}\ \Lambda)$ such that

$$\Gamma \simeq \mathrm{End}_{\mathrm{K^b}(\mathrm{proj}\ \Lambda)}(T).$$

Let $T = T_1 \oplus \ldots \oplus T_j \oplus \ldots \oplus T_n$ be a basic object in a subclass $\mathcal{C}$ of an additive category. If we can replace a direct summand T_j by $T_j^* \ (\not\cong T_j)$ via certain procedure to get a new object

$$\mu_j(T) = T_1 \oplus \ldots \oplus T_j^* \oplus \ldots \oplus T_n,$$

so that $\mu_j(T)$ also lies in $\mathcal{C}$, then $\mu_j(T)$ is called the mutation of T with respect to T_j.

It is known that the mutation at an indecomposable direct summand of tilting objects is not always possible. To complete this, Aihara and Iyama introduced silting mutation of silting complexes in 2012, which is currently known as Silting Theory.

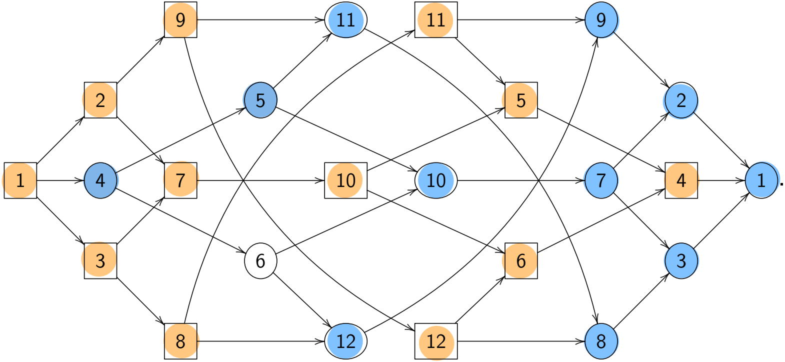
After years of research, it is seen that

- the set of silting complexes admits a partial order, such that its Hasse quiver realizes the left/right silting mutation.
- the set of two-term silting complexes is in bijection with the set of support τ-tilting modules which are introduced by Adachi, Iyama and Reiten in 2014.
- etc.

In this talk, we explain a symmetry for the Hasse quiver of the poset of two-term silting complexes.

An example

The Hasse quiver of the preprojective algebra of Dynkin type $\mathbb{A}_3$ is displayed as



.

Silting Theory

We fix some symbols as follows.

- $\mathcal{K}_\Lambda := \mathsf{K}^{\mathsf{b}}(\operatorname{proj} \Lambda)$: the homotopy category of bounded complexes of finitely generated projective Λ-modules
- thick T: the smallest thick subcategory of $\mathcal{K}_\Lambda$ containing T, for any $T \in \mathcal{K}_\Lambda$
- add(T): the full subcategory of $\mathcal{K}_\Lambda$ whose objects are direct summands of finite direct sums of copies of T, for any $T \in \mathcal{K}_\Lambda$

Definition 1.1 (Aihara-Iyama, 2012)

Let $T \in \mathcal{K}_\Lambda$. Then,

(1) T is called presilting if $\operatorname{Hom}_{\mathcal{K}_\Lambda}(T, T[i]) = 0$ for any $i > 0$.

(2) T is called silting if T is presilting and thick $T = \mathcal{K}_\Lambda$.

(3) T is called tilting if T is silting and $\operatorname{Hom}_{\mathcal{K}_\Lambda}(T, T[i]) = 0$ for any $i < 0$.

A partial order on silt Λ

We denote by silt Λ the set of isomorphism classes of basic silting complexes in $\mathcal{K}_\Lambda$.

Definition 1.2 (Aihara-Iyama, 2012)

For any $T, S \in$ silt Λ, we say $T \geq S$ if

$$\mathrm{Hom}_{\mathcal{K}_\Lambda}(T, S[i]) = 0$$

for any $i > 0$.

Then, $\geq$ gives a partial order on the set silt Λ.

Silting mutation

Theorem-Definition 1.3 (Aihara-Iyama, 2012)

For any $S, T \in \operatorname{silt}\Lambda$, the following conditions are equivalent.

(1) S is a left mutation of T.

(2) T is a right mutation of S.

(3) $T > S$ and there is no $X \in \operatorname{silt}\Lambda$ such that $T > X > S$.

Let $T = T_1 \oplus \cdots \oplus T_j \oplus \cdots \oplus T_n \in \mathsf{silt}\,\Lambda$ with an indecomposable direct summand T_j. Take a minimal left $\mathsf{add}(T/T_j)$-approximation π and a triangle

$$T_j \xrightarrow{\pi} Z \longrightarrow \mathsf{cone}(\pi) \longrightarrow T_j[1],$$

where $\mathsf{cone}(\pi)$ is the mapping cone of π. Then, $\mathsf{cone}(\pi)$ is indecomposable and $\mu_j^-(T) := \mathsf{cone}(\pi) \oplus (T/T_j)$ is again a basic silting complex in $\mathcal{K}_\Lambda$.

We call $\mu_j^-(T)$ the irreducible left silting mutation of T with respect to T_j, or simply, the left mutation of T with respect to T_j.

Dually, we define the irreducible right silting mutation $\mu_j^+(T)$ of T with respect to T_j.

Two-term silting complexes

A complex in $\mathcal{K}_\Lambda$ is called two-term if it is homotopy equivalent to a complex T, which is concentrated in degrees 0 and -1, i.e.,

$$T = (T^{-1} \xrightarrow{d_T^{-1}} T^0) := \\ \cdots \longrightarrow 0 \longrightarrow T^{-1} \xrightarrow{d_T^{-1}} T^0 \longrightarrow 0 \longrightarrow \cdots .$$

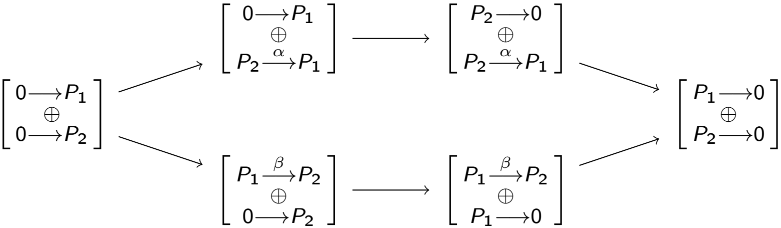
We denote by 2-silt Λ the subset of two-term complexes in silt Λ.

Obviously, 2-silt Λ is a poset under the partial order $\geq$ on silt Λ.

We denote by $\mathcal{H}$(2-silt Λ) the Hasse quiver of 2-silt Λ, which is compatible with the left/right mutation of two-term silting complexes.

Example

Let $\Lambda = K(\ 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2\ )/\langle \alpha\beta, \beta\alpha \rangle$. The Hasse quiver $\mathcal{H}$(2-silt Λ) is displayed by



.

Proposition 1.4 (Aihara-Iyama, 2012)

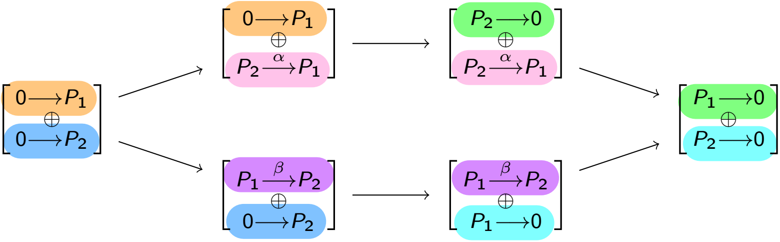
Let $T = (T^{-1} \to T^0) \in$ 2-silt Λ. Then, add $\Lambda =$ add $(T^0 \oplus T^{-1})$ and add $T^0 \cap$ add $T^{-1} = 0$.

Proposition 1.5 (Adachi-Iyama-Reiten, 2014)

Let T be a two-term presilting complex in $\mathcal{K}_\Lambda$ with $|T| = |\Lambda| - 1$. Then, T is a direct summand of exactly two basic two-term silting complexes in 2-silt Λ.

Example

Let $\Lambda = K(\, 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \,)/\langle \alpha\beta, \beta\alpha \rangle$. The Hasse quiver $\mathcal{H}(\text{2-silt } \Lambda)$ is displayed by



.

g-vectors

Let $|\Lambda| = n$ with $P_1, \ldots, P_n$ as indecomposable projective modules. The isomorphism classes $[P_1], [P_2], \ldots, [P_n]$ of indecomposable complexes in $\mathcal{K}_\Lambda$ concentrated in degree 0, form a K-basis of the Grothendieck group $K_0(\mathcal{K}_\Lambda)$. If a two-term complex T in $\mathcal{K}_\Lambda$ is written as

$$\left(\bigoplus_{i=1}^{n} P_i^{\oplus b_i} \longrightarrow \bigoplus_{i=1}^{n} P_i^{\oplus a_i} \right),$$

then the class $[T]$ can be identified by an integer vector

$$g(T) = (a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n) \in \mathbb{Z}^n,$$

which is called the g-vector of T.

Proposition 1.6 (Adachi-Iyama-Reiten, 2014)

Let $T \in$ 2-silt Λ. Then, the map $T \mapsto g(T)$ is an injection.

Symmetry on two-term silting quivers

Let $(-)^* := \mathrm{Hom}_?(-, \Lambda)$ for $? = \mathcal{K}_\Lambda$ or $\mathcal{K}_{\Lambda^{\mathrm{op}}}$. For any $T \in$ 2-silt Λ,

$$T = \left(0 \longrightarrow \bigoplus_{i \in I} P_i^{\oplus a_i} \xrightarrow{d} \bigoplus_{j \in J} P_j^{\oplus a_j} \longrightarrow 0 \right),$$

with $I \cap J = 0$ and $I \cup J = \{1, 2, \cdots, n\}$. Then,

$$T^* = \left(0 \longrightarrow 0 \longrightarrow \bigoplus_{j \in J} (P_j^*)^{\oplus a_j} \xrightarrow{d^*} \bigoplus_{i \in I} (P_i^*)^{\oplus a_i} \right).$$

If there is an algebra isomorphism $\sigma : \Lambda^{\mathrm{op}} \to \Lambda$, then σ induces a permutation on $\{1, 2, \cdots, n\}$ by $\sigma(e_i^*) = e_j$. We then obtain an equivalence $\mathcal{K}_{\Lambda^{\mathrm{op}}} \to \mathcal{K}_\Lambda$, also denoted by σ.

We have

$$\sigma(T^*) = \left(0 \longrightarrow 0 \longrightarrow \bigoplus_{j\in J} (P_{\sigma(j)})^{\oplus a_{\sigma(j)}} \xrightarrow{\sigma(d^*)} \bigoplus_{i\in I} (P_{\sigma(i)})^{\oplus a_{\sigma(i)}} \right),$$

which is again a silting complex in $\mathcal{K}_\Lambda$. Set $S_\sigma := [1] \circ \sigma \circ (-)^*$.

Theorem 2.1 (Aihara-W, 2022)

The functor S_σ induces an anti-automorphism of the poset 2-silt Λ. For any two-term silting complex T with $g(T) = (a_1, a_2, \cdots, a_n)$, we have

$$g(S_\sigma(T)) = -(a_{\sigma(1)}, a_{\sigma(2)}, \cdots, a_{\sigma(n)}).$$

Let $\sigma : \Lambda^{\mathrm{op}} \to \Lambda$ be an algebra isomorphism.

- σ fixes a primitive idempotent e, i.e., $\sigma(P^*) = P$;
- σ fixes no primitive idempotent, see an example later.

Let P be an indecomposable projective Λ-module. We define two subsets of 2-silt Λ by

$$\begin{aligned} \mathcal{T}_P^- &:= \{T \in \text{2-silt}\,\Lambda \mid \mu_P^-(\Lambda) \geq T \geq \Lambda[1]\} \text{ and} \\ \mathcal{T}_P^+ &:= \{T \in \text{2-silt}\,\Lambda \mid \Lambda \geq T \geq \mu_{P[1]}^+(\Lambda[1])\}. \end{aligned}$$

Lemma 2.2

We have $\mathcal{T}_P^- \sqcup \mathcal{T}_P^+ = \text{2-silt}\,\Lambda$.

Main result

For any $T \in \mathcal{T}_P^- = \{T \in \text{2-silt}\,\Lambda \mid \mu_P^-(\Lambda) \geq T \geq \Lambda[1]\}$, we have

$$\Lambda \geq S_\sigma(T) \geq \mu^+_{\sigma(P^*)[1]}(\Lambda),$$

i.e., $S_\sigma(T) \in \mathcal{T}^+_{\sigma(P^*)}$. Thus,

Theorem 2.3 (Aihara-W, 2022)

If $\sigma(P^*) = P$, then S_σ gives a bijection between $\mathcal{T}_P^-$ and $\mathcal{T}_P^+$. In particular, the cardinality of 2-silt Λ is the double of that of $\mathcal{T}_P^-$.

We then apply Theorem 2.3 to some classes of algebras.

Opposite quivers

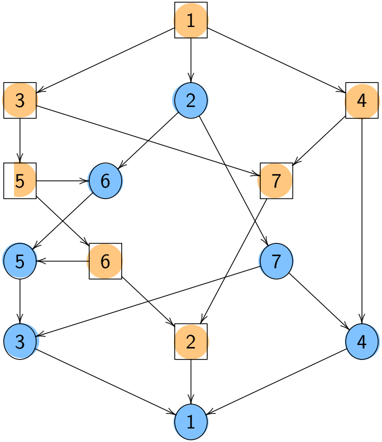
The opposite quiver Q^{op} of a finite quiver Q consists of the same vertices as Q and reversed arrows a^* for arrows a of Q. For an admissible ideal I of KQ, reversing arrows gives the admissible ideal I^{op} of KQ^{op}; for example, $ab \in I$ implies $b^*a^* \in I^{\mathrm{op}}$.

Proposition 2.4

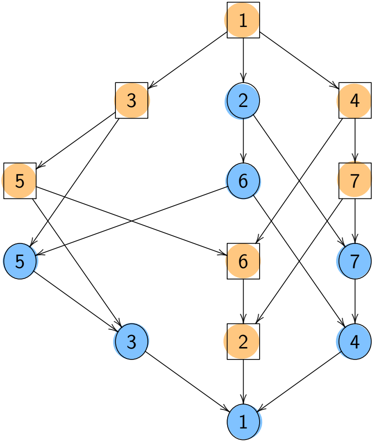
Let $\Lambda = KQ/I$. If there is an isomorphism $\iota : Q^{\mathrm{op}} \to Q$ satisfying $I^{\mathrm{op}} = \iota^{-1}(I)$ and fixing a vertex i, put $P_i = e_i\Lambda$, then there is a bijection between $\mathcal{T}^{-}_{P_i}$ and $\mathcal{T}^{+}_{P_i}$.

For example, $\Lambda = KQ/I$ with $I = \langle x^r \rangle$ and

- $Q = A_{2n+1} : 1 \xrightarrow{x} 2 \xrightarrow{x} \cdots \xrightarrow{x} (2n+1)$;
- $Q = \widetilde{A}_n :\ 1 \xrightarrow[x]{} 2 \xrightarrow[x]{} \cdots \xrightarrow[x]{} n$, with an arrow $x : n \to 1$.



$\mathcal{H}(\text{2-silt } KA_3)$



$\mathcal{H}(\text{2-silt } K\widetilde{A}_3/\langle x^2\rangle)$

Double quivers

The double quiver $\overline{Q}$ of $Q = (Q_0, Q_1)$ is given by $\overline{Q}_0 := Q_0$ and

$$\overline{Q}_1 := Q_1 \sqcup \{a^* \mid a \in Q_1\}.$$

The assignments $v \mapsto v$ $(v \in Q_0)$, $a^* \mapsto a^*$, $(a^*)^* \mapsto a$ $(a \in Q_1)$ give an isomorphism $\iota : \overline{Q}^{\mathrm{op}} \to \overline{Q}$; note that ι fixes all vertices.

Proposition 2.5

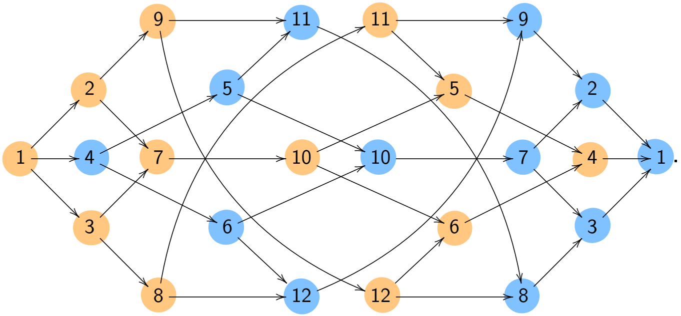
Let Λ be an algebra presented by a double quiver.

- $S_\sigma(\mathcal{T}_P^-) = \mathcal{T}_P^+$ for any indecomposable projective module P.
- For any $T \in$ 2-silt Λ, we have $g(S_\sigma(T)) = -g(T)$.

For example,

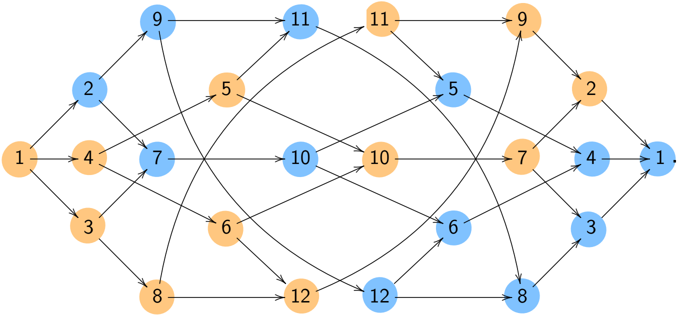
- preprojective algebras of Dynkin quivers;
- symmetric algebras with radical cube zero, including multiplicity-free Brauer line/cycle algebras;
- quasi-hereditary algebras with a duality, including all q-Schur algebras.

The preprojective algebra of Dynkin type $\mathbb{A}_3$



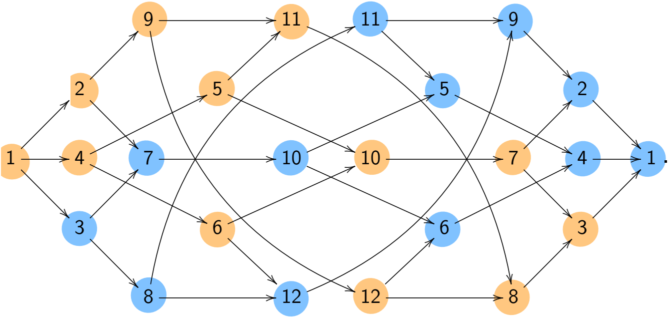
$\mathcal{T}_{P_2}^{-}$ and $\mathcal{T}_{P_2}^{+}$

The preprojective algebra of Dynkin type $\mathbb{A}_3$



$\mathcal{T}^-_{P_1}$ and $\mathcal{T}^+_{P_1}$

The preprojective algebra of Dynkin type $\mathbb{A}_3$



$\mathcal{T}^{-}_{P_3}$ and $\mathcal{T}^{+}_{P_3}$

Applying the main result twice

Let $\sigma : \Lambda^{op} \to \Lambda$ be an algebra isomorphism fixing a primitive idempotent e of Λ; set $P := e\Lambda$.

Suppose that $\mu_P^-(\Lambda) = P' \oplus (\Lambda/P)$ and $\Gamma := \mathrm{End}_{\mathcal{K}_\Lambda}(\mu_P^-(\Lambda))$, e' denotes the idempotent of Γ corresponding to P'.

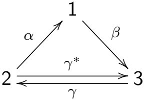
Proposition 2.6

If $\mu_P^-(\Lambda)$ is tilting, and there exists an algebra isomorphism $\sigma' : \Gamma^{op} \to \Gamma$ satisfying $\sigma'((e')^*) = e'$, then 2-silt Λ and 2-silt Γ have the same cardinality.

Sketch of the proof: Let $T \in$ 2-silt Λ. Then,

$$\begin{aligned}&\{T \mid \mu_P^-(\Lambda) \geq T \geq \mu_P^-(\Lambda)[1]\}\\ &= \{T \mid \mu_{P'}^-\mu_P^-(\Lambda) \geq T \geq \mu_P^-(\Lambda)[1]\} \sqcup \{T \mid \mu_P^-(\Lambda) \geq T \geq \Lambda[1]\}\end{aligned}$$

Let Λ be the algebra presented by



$$\text{with } \begin{cases} \beta\gamma\alpha = 0 = \gamma(\gamma^*\gamma)^3 \\ \alpha\beta = \gamma^*\gamma\gamma^* \end{cases} .$$

Then, Λ is symmetric and there exists an algebra iso. $\sigma : \Lambda^{\mathrm{op}} \to \Lambda$ fixing the vertex 1 and switches the vertices 2 and 3. Set $P_1 := e_1\Lambda$.

The endomorphism algebra $\Gamma = \mathrm{End}_{\mathcal{K}_\Lambda}(\mu^-_{P_1}(\Lambda))$ is given by

$$2 \underset{\alpha^*}{\overset{\alpha}{\rightleftarrows}} 1 \underset{\beta^*}{\overset{\beta}{\rightleftarrows}} 3 \ \text{ with } \begin{cases} \alpha\beta\beta^*\beta = \beta^*\beta\beta^*\alpha^* = \alpha\alpha^* = 0 \\ \alpha^*\alpha = (\beta\beta^*)^2 \end{cases} .$$

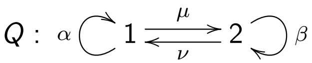
It is seen that there exists an algebra isomorphism $\sigma' : \Gamma^{\mathrm{op}} \to \Gamma$ fixing the vertex 1.

It turns out that $\#\text{2-silt}\,\Lambda = \#\text{2-silt}\,\Gamma = 32$.

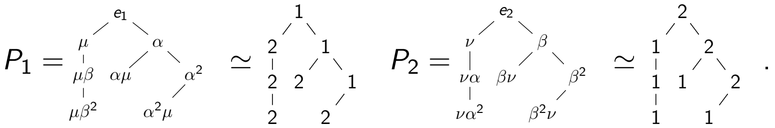
Application to two-point algebras

An example

Let $\Lambda = KQ/I$ be the monomial algebra presented by



and $I = \langle \alpha^3, \beta^3, \mu\nu, \nu\mu, \alpha\mu\beta, \beta\nu\alpha, \nu\alpha\mu, \mu\beta\nu, \nu\alpha^2\mu, \mu\beta^2\nu \rangle$. The indecomposable projective Λ-modules are



There exist two algebra isomorphisms $\sigma, \sigma' : \Lambda^{\mathrm{op}} \to \Lambda$ satisfying

$$\sigma(e_1^*) = e_2,\ \sigma(e_2^*) = e_1 \text{ and } \sigma'(e_1^*) = e_1,\ \sigma'(e_2^*) = e_2.$$

By direct calculation, we find the following chain T in $\mathcal{H}$(2-silt Λ),

$$\begin{bmatrix} 0 \longrightarrow P_1 \\ \oplus \\ 0 \longrightarrow P_2 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 \longrightarrow P_1 \\ \oplus \\ P_2 \xrightarrow{f_1} P_1^{\oplus 3} \end{bmatrix} \longrightarrow \begin{bmatrix} P_2 \xrightarrow{f_2} P_1^{\oplus 2} \\ \oplus \\ P_2 \xrightarrow{f_1} P_1^{\oplus 3} \end{bmatrix} \longrightarrow \begin{bmatrix} P_2 \xrightarrow{f_2} P_1^{\oplus 2} \\ \oplus \\ P_2^{\oplus 2} \xrightarrow{f_3} P_1^{\oplus 3} \end{bmatrix} \longrightarrow \begin{bmatrix} P_2 \xrightarrow{\mu} P_1 \\ \oplus \\ P_2^{\oplus 2} \xrightarrow{f_3} P_1^{\oplus 3} \end{bmatrix}$$

where $f_1 = \begin{pmatrix} \mu \\ \mu\beta \\ \mu\beta^2 \end{pmatrix}$, $f_2 = \begin{pmatrix} \mu \\ \mu\beta \end{pmatrix}$, $f_3 = \begin{pmatrix} \mu & 0 \\ -\mu\beta & \mu \\ 0 & \mu\beta \end{pmatrix}$. Then, the g-vectors of T are displayed as

$$\begin{matrix} (1,0) \\ \oplus \\ (0,1) \end{matrix} \longrightarrow \begin{matrix} (1,0) \\ \oplus \\ (3,-1) \end{matrix} \longrightarrow \begin{matrix} (2,-1) \\ \oplus \\ (3,-1) \end{matrix} \longrightarrow \begin{matrix} (2,-1) \\ \oplus \\ (3,-2) \end{matrix} \longrightarrow \begin{matrix} (1,-1) \\ \oplus \\ (3,-2) \end{matrix}.$$

By Theorem 2.1, there are other left mutation chains

$$S_\sigma(\mathrm{T}),\ S_{\sigma'}(\mathrm{T}),\ S_{\sigma'}(S_\sigma(\mathrm{T})) = S_\sigma(S_{\sigma'}(\mathrm{T})) \in \mathcal{H}(\text{2-silt } \Lambda)$$

whose g-vectors are displayed as follows.

$$g(\mathsf{T}) : \quad \begin{matrix}(1,0)\\ \oplus \\ (0,1)\end{matrix} \longrightarrow \begin{matrix}(1,0)\\ \oplus \\ (3,-1)\end{matrix} \longrightarrow \begin{matrix}(2,-1)\\ \oplus \\ (3,-1)\end{matrix} \longrightarrow \begin{matrix}(2,-1)\\ \oplus \\ (3,-2)\end{matrix} \longrightarrow \begin{matrix}(1,-1)\\ \oplus \\ (3,-2)\end{matrix}.$$

$g(S_\sigma(\mathsf{T}))$:

$$\begin{matrix}(1,-1)\\ \oplus \\ (2,-3)\end{matrix} \longrightarrow \begin{matrix}(1,-2)\\ \oplus \\ (2,-3)\end{matrix} \longrightarrow \begin{matrix}(1,-2)\\ \oplus \\ (1,-3)\end{matrix} \longrightarrow \begin{matrix}(0,-1)\\ \oplus \\ (1,-3)\end{matrix} \longrightarrow \begin{matrix}(0,-1)\\ \oplus \\ (-1,0)\end{matrix}.$$

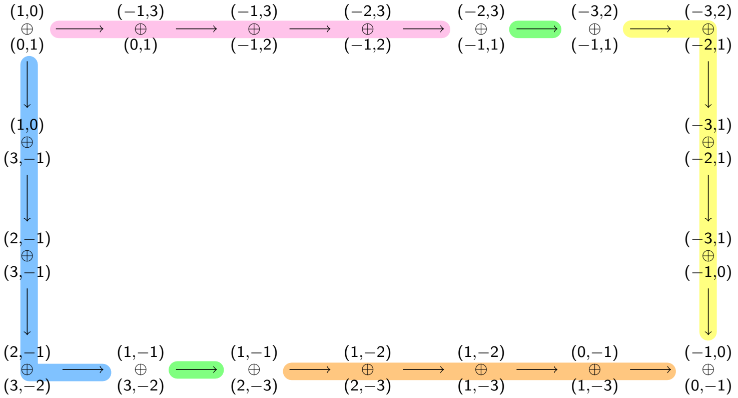
$g(S_{\sigma'}(\mathsf{T}))$:

$$\begin{matrix}(-1,1)\\ \oplus \\ (-3,2)\end{matrix} \longrightarrow \begin{matrix}(-2,1)\\ \oplus \\ (-3,2)\end{matrix} \longrightarrow \begin{matrix}(-2,1)\\ \oplus \\ (-3,1)\end{matrix} \longrightarrow \begin{matrix}(-1,0)\\ \oplus \\ (-3,1)\end{matrix} \longrightarrow \begin{matrix}(-1,0)\\ \oplus \\ (0,-1)\end{matrix}.$$

$g(S_{\sigma'}(S_\sigma(\mathsf{T}))) = g(S_\sigma(S_{\sigma'}(\mathsf{T})))$:

$$\begin{matrix}(1,0)\\ \oplus \\ (0,1)\end{matrix} \longrightarrow \begin{matrix}(-1,3)\\ \oplus \\ (0,1)\end{matrix} \longrightarrow \begin{matrix}(-1,3)\\ \oplus \\ (-1,2)\end{matrix} \longrightarrow \begin{matrix}(-2,3)\\ \oplus \\ (-1,2)\end{matrix} \longrightarrow \begin{matrix}(-2,3)\\ \oplus \\ (-1,1)\end{matrix}.$$

By Proposition 1.5, each indecomposable two-term presilting complex in $\mathcal{K}_\Lambda$ is a direct summand of exactly two two-term silting complexes in 2-silt Λ. The g-vectors for Λ must be given by



.

References

[AA] T. Adachi and T. Aoki, The number of two-term tilting complexes over symmetric algebras with radical cube zero. Preprint (2018), arXiv: 1805.08392.

[AI] T. Aihara and O. Iyama, Silting mutation in triangulated categories. *J. Lond. Math. Soc. (2)* **85** (2012), no. 3, 633–668.

[AIR] T. Adachi, O. Iyama and I. Reiten, τ-tilting theory. *Compos. Math.* **150** (2014), no. 3, 415–452.

[Ric] J. Rickard, Morita theory for derived categories. *J. London Math. Soc.*, **39** (1989), no. 3, 436–456.

Thank you very much for your attention !