On τ-tilting finiteness of Schur algebras[1]

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Outline

Introduction

Let A be a finite-dimensional algebra over an algebraically closed field K. As a generalization of the classical tilting modules, Adachi, Iyama and Reiten introduced the support τ-tilting modules, which have many nice properties. For example,

- these modules are bijectively corresponding to two-term silting complexes, left finite semibricks and so on.
- there is a partial order on the set of isomorphism classes of basic support τ-tilting modules such that its Hasse quiver realizes the left mutation. (See Proposition 1.4 in this talk.)

We recall that A is τ-tilting finite if there are only finitely many pairwise non-isomorphic basic support τ-tilting A-modules. Otherwise, A is called τ-tilting infinite.

Background

(1) The τ-tilting finiteness for several classes of algebras is known, such as

- preprojective algebras of Dynkin type (Mizuno, 2014);
- algebras with radical square zero (Adachi, 2016);
- Brauer graph algebras (Adachi-Aihara-Chan, 2018);
- biserial algebras (Mousavand, 2019);
- minimal wild two-point algebras (W, 2019).

(2) Representation-finite algebras are τ-tilting finite and the converse is not true in general. For example, let

$$\Lambda_n := K(\circ \xrightarrow{\alpha} \circ \circlearrowleft \beta)/ < \beta^n, \alpha\beta^2 >, \ n \geqslant 2,$$

then Λ_n is τ-tilting finite (by direct calculation). But Λ_n is representation-finite if $n \leqslant 5$; tame if $n = 6$; wild if $n \geqslant 7$.

However, the converse is true in some special cases, including

- cycle-finite algebras (Malicki-Skowroński, 2016);
- gentle algebras (Plamondon, 2018);
- tilted and cluster-tilted algebras (Zito, 2019);
- simply connected algebras (W, 2019);
- quasi-tilted algebras, locally hereditary algebras, etc., (Aihara-Honma-Miyamoto-W, 2020).

The subject of this talk

Let n, r be positive integers and $\mathbb{F}$ an algebraically closed field of characteristic p. We take an n-dimensional vector space V over $\mathbb{F}$ with a basis $\{v_1, v_2, \ldots, v_n\}$. Then, the r-fold tensor product $V^{\otimes r} := V \otimes_{\mathbb{F}} V \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} V$ has a $\mathbb{F}$-basis given by

$$\{v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r} \mid 1 \leqslant i_j \leqslant n \text{ for all } 1 \leqslant j \leqslant r\}.$$

Let G_r be the symmetric group on r symbols. Then,

$$(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r}) \cdot \sigma = v_{i_{\sigma(1)}} \otimes v_{i_{\sigma(2)}} \otimes \cdots \otimes v_{i_{\sigma(r)}}.$$

for any $\sigma \in G_r$. We define the Schur algebra $S(n, r)$ by the endomorphism ring $\mathrm{End}_{\mathbb{F}G_r}(V^{\otimes r})$.

In this talk, we will discuss the τ-tilting finiteness of $S(n, r)$, and see when the condition

$$\tau\text{-titling finite} \Leftrightarrow \text{representation-finite}$$

is true in the class of Schur algebras.

τ-tilting theory

Auslander-Reiten translation

Let A be a finite-dimensional basic algebra over an algebraically closed field K. For an A-module M with a minimal projective presentation

$$P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0,$$

the transpose of M is given by

$$\mathrm{Tr}\ M := \mathsf{coker}\ \mathsf{Hom}_A(d_1, A).$$

Then, the Auslander-Reiten translation is defined by

$$\tau(M) := D\mathrm{Tr}\ M,$$

where $D = \mathsf{Hom}_K(-, K)$ is the standard K-duality.

Let $|M|$ be the number of isomorphism classes of indecomposable direct summands of M.

Definition 1.1 (Adachi-Iyama-Reiten, 2014)

Let M be a right A-module and P a projective A-module.

(1) M is called τ-rigid if $\mathrm{Hom}_A(M, \tau M) = 0$.

(2) M is called τ-tilting if M is τ-rigid and $|M| = |A|$.

(3) M is called support τ-tilting if there exists an idempotent e of A such that M is a τ-tilting (A/AeA)-module.

We denote by τ-rigid A (resp., sτ-tilt A) the set of isomorphism classes of indecomposable τ-rigid (resp., basic support τ-tilting) A-modules.

Example

Let $A := K(\, 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \,)/ < \alpha\beta, \beta\alpha >$. We denote by S_i the simple A-modules and P_i the indecomposable projective A-modules. Then, we have

$$\tau(S_1) = S_2, \tau(S_2) = S_1, \tau(P_1) = 0, \tau(P_2) = 0.$$

Thus,

- τ-rigid $A = \{P_1, P_2, S_1, S_2, 0\}$;
- sτ-tilt $A = \{P_1 \oplus P_2, P_1 \oplus S_1, S_2 \oplus P_2, S_1, S_2, 0\}$;
- $P_1 \oplus P_2, P_1 \oplus S_1$ and $S_2 \oplus P_2$ are τ-tilting modules.

Mutation

We denote by add(M) (resp., Fac(M)) the full subcategory whose objects are direct summands (resp., factor modules) of finite direct sums of copies of M.

Definition 1.2 (Adachi-Iyama-Reiten, 2014)

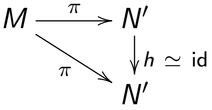
Let $T = M \oplus N$ be a basic support τ-tilting A-module with an indecomposable direct summand M satisfying $M \notin \mathsf{Fac}(N)$. We take a minimal left add(N)-approximation π with an exact sequence

$$M \xrightarrow{\pi} N' \longrightarrow \mathsf{coker}\ \pi \longrightarrow 0.$$

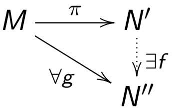
Then, we call $\mu_M^-(T) := \mathsf{coker}\ \pi \oplus N$ the left mutation of T with respect to M, which is again a basic support τ-tilting A-module.

A morphism $\pi : M \to N'$ is called a minimal left add(N)-approximation of M if $N' \in \mathrm{add}(N)$ and it satisfies:

(1) any $h : N' \to N'$ satisfying $h \circ \pi = \pi$ is an automorphism.



(2) for any $N'' \in \mathrm{add}(N)$ and $g : M \to N''$, there exists $f : N' \to N''$ such that $f \circ \pi = g$.



Example

Let $A = K(\ 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2\ )/ < \alpha\beta, \beta\alpha >$, then $P_1 \oplus P_2$ is a τ-tilting module. We consider the left mutation with respect to P_2,

$$P_2 \xrightarrow{\pi} P_1 \longrightarrow \text{coker } \pi \longrightarrow 0,$$

where $\pi : {}^{e_2}_{\beta} \xrightarrow{\alpha\cdot} {}^{e_1}_{\alpha}$ is a minimal left add(P_1)-approximation. Then,

$$\text{coker } \pi = S_1 \text{ and } \mu^-_{P_2}(A) = P_1 \oplus S_1.$$

In fact, we have the following mutation quiver of sτ-tilt A.

$$\begin{array}{ccccc} P_1 \oplus P_2 & \longrightarrow & P_1 \oplus S_1 & \longrightarrow & S_1 \\ \downarrow & & & & \downarrow \\ S_2 \oplus P_2 & \longrightarrow & S_2 & \longrightarrow & 0 \end{array}$$

Poset structure on sτ-tilt A

Recall that Fac(M) is the full subcategory whose objects are factor modules of finite direct sums of copies of M.

Definition 1.3 (Adachi-Iyama-Reiten, 2014)

For $M, N \in$ sτ-tilt A, we say $M \geqslant N$ if $\mathrm{Fac}(N) \subseteq \mathrm{Fac}(M)$.

Example

Let A be the previous algebra. The Hasse quiver of sτ-tilt A is

$$\begin{array}{ccccc} P_1 \oplus P_2 & \xrightarrow{>} & P_1 \oplus S_1 & \xrightarrow{>} & S_1 \\ \downarrow{\scriptstyle >} & & & & \downarrow{\scriptstyle >} \\ S_2 \oplus P_2 & \xrightarrow{>} & S_2 & \xrightarrow{>} & 0 \end{array}$$

Proposition 1.4 (Adachi-Iyama-Reiten, 2014)

The mutation quiver $\mathcal{Q}(\mathsf{s}\tau\text{-tilt } A)$ and the Hasse quiver $\mathcal{H}(\mathsf{s}\tau\text{-tilt } A)$ coincide.

Proposition 1.5 (Adachi-Iyama-Reiten, 2014)

If the Hasse quiver $\mathcal{H}(\mathsf{s}\tau\text{-tilt } A)$ contains a finite connected component Δ, then $\mathcal{H}(\mathsf{s}\tau\text{-tilt } A) \simeq \Delta$.

Reduction theorems

Proposition 1.6 (Demonet-Iyama-Jasso, 2017)

If A is τ-tilting finite, then

(1) the quotient A/I is τ-tilting finite for any two-sided ideal I of A.

(2) the idempotent truncation eAe is τ-tilting finite for any idempotent e of A.

Proposition 1.7 (Eisele-Janssens-Raedschelders, 2018)

Let I be a two-sided ideal generated by central elements which are contained in the Jacobson radical of A. Then, there exists a poset isomorphism between sτ-tilt A and sτ-tilt (A/I).

Schur algebras

Let $\mathbb{F}$ be an algebraically closed field, char $\mathbb{F} = 0$ or p (a prime).

- r: a positive number.
- $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$: a partition of r with at most n parts.
- $\Omega(n, r)$: the set of all partitions of r with at most n parts.
- G_r: the symmetric group on r symbols.
- $\mathbb{F}G_r$: the group algebra of G_r over $\mathbb{F}$.

We recall that the Schur algebra $S(n, r) = \mathrm{End}_{\mathbb{F}G_r}(V^{\otimes r})$ is not necessarily basic. Also, it is not necessarily connected. Thus, our aim in this section is to find the basic algebra of $S(n, r)$. In other words, we need to find all indecomposable pairwise non-isomorphic direct summands of $V^{\otimes r}$.

Permutation modules

We may define the Young subgroup G_λ of G_r associated with a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ of r by

$$G_\lambda := G_{\lambda_1} \times G_{\lambda_2} \times \ldots \times G_{\lambda_n}.$$

Then, the permutation $\mathbb{F}G_r$-module M^λ is $1_{G_\lambda} \uparrow^{G_r}$, where 1_{G_λ} denotes the trivial module for G_λ and $\uparrow$ denotes the induction.

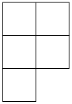
(Note that M^λ is not necessarily indecomposable.) Then, M^λ with $\lambda \in \Omega(n,r)$ can be regarded as direct summands of $V^{\otimes r}$, i.e.,

$$V^{\otimes r} \simeq \bigoplus_{\lambda \in \Omega(n,r)} n_\lambda M^\lambda,$$

with multiplicities n_λ.

Specht modules and the decomposition matrix

Let $[\lambda]$ be the Young diagram of partition λ. We call λ a *p*-regular partition if no p rows of $[\lambda]$ have the same length. Otherwise, λ is called p-singular. For example,

the partition $[(2,2,1)] =$  is $\begin{cases} \text{2-singular} \\ \text{3-regular} \end{cases}$.

Let S^λ be the Specht module of $\mathbb{F}G_r$ corresponding to λ.

- If char $\mathbb{F} = 0$, then $\left\{ S^\lambda \mid \lambda : \text{ partition of } r \right\}$ is a complete set of pairwise non-isomorphic simple $\mathbb{F}G_r$-modules.
- If char $\mathbb{F} = p$, then $\left\{ D^\lambda \mid \lambda : \ p\text{-regular partition of } r \right\}$ is a complete set of pairwise non-isomorphic simple $\mathbb{F}G_r$-modules, where D^λ is the unique simple top $S^\lambda/(\text{rad } S^\lambda)$ of S^λ with λ being p-regular.

(Cont'd) In the case of a p-singular partition μ, all of the composition factors of S^μ are D^λ such that λ is a p-regular partition which dominates μ. Therefore, the decomposition matrix of $\mathbb{F}G_r$ has the following form (see [James, 1978]),

$$\begin{array}{r} \\ S^\lambda, \lambda \ p\text{-regular} \left\{ \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \right. \\ \\ S^\lambda, \lambda \ p\text{-singular} \left\{ \begin{array}{c} \\ \\ \end{array} \right. \end{array} \overbrace{\begin{pmatrix} 1 & & & & \\ * & 1 & & O & \\ * & * & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ * & * & * & \dots & 1 \\ - & - & - & - & - \\ * & * & * & \dots & * \\ * & * & * & \dots & * \end{pmatrix}}^{D^\mu, \mu \ p\text{-regular}}.$$

Young modules

Let $\mathbb{F}$ be an algebraically closed field and p a prime.

- χ^λ: the (ordinary) irreducible character of G_r corresponding to λ over char $\mathbb{F} = 0$. ($\xleftrightarrow{1:1}$ Specht module S^λ)
- ch M^λ: the associated ordinary character of permutation module M^λ over char $\mathbb{F} = p$.

Then,

$$\text{ch } M^\lambda = \chi^\lambda + \sum_{\mu \rhd \lambda} k_\mu \chi^\mu$$

with multiplicities k_μ (which can be zero).

Let $M^\lambda := \oplus_{i=1}^n Y_i$ with $n \in \mathbb{N}$. The unique direct summand Y_i which the ordinary character χ^λ occurs in ch Y_i, is called the Young module corresponding to λ and we denote it by Y^λ.

Proposition 2.1 (Martin, 1993)

The set $\{ Y^\lambda \mid \lambda \in \Omega(n,r) \}$ is a complete set of pairwise non-isomorphic Young modules which occurs as indecomposable direct summands of permutation modules M^λ with $\lambda \in \Omega(n,r)$.

Let Y^λ be the Young module corresponding to λ.

- $Y^\lambda = (Y^\lambda)^*$ with $(-)^* := \mathrm{Hom}(-, \mathbb{F})$.
- Y^λ has a Specht filtration: $Y^\lambda = Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_k = 0$ for some $k \in \mathbb{N}$ with each Z_i/Z_{i+1} isomorphic to a Specht module S^μ with $\mu \trianglerighteq \lambda$.
- If λ is a partition with at most n parts, then each composition factor D^μ of Y^λ is also corresponding to the partition μ with at most n parts.

The basic algebra of $S(n, r)$

Let B be a block of $\mathbb{F}G_r$ labeled by a p-core ω, it is well-known that a partition λ belongs to B if and only if λ has the same p-core ω. Then, we define

$$S_B := \mathrm{End}_{\mathbb{F}G_r}\left(\bigoplus_{\lambda\in B\cap\Omega(n,r)} Y^{\lambda}\right)$$

and the basic algebra of $S(n, r)$ is $\bigoplus S_B$, where the sum is taken over all blocks of $\mathbb{F}G_r$. Moreover, S_B is a direct sum of blocks of the basic algebra of $S(n, r)$.

Next, our aim is to find the quiver and relations of S_B, i.e.,

$$S_B = \mathbb{F}Q/I.$$

The quiver of the basic algebra of $S(2, r)$

We recall that the vertex in the quiver is labeled by

$$Y^{(\lambda_1,\lambda_2)} \xleftrightarrow{1:1} (\lambda_1, \lambda_2) \xleftrightarrow{1:1} v^s,$$

where $s := \lambda_1 - \lambda_2$ is uniquely determined based on $\lambda_1 + \lambda_2 = r$. Let $n(v^s, v^t)$ be the number of arrows from v^s to v^t. We have $n(v^s, v^t) = n(v^t, v^s)$. Thus, we suppose $s > t$.

Theorem 2.2 (Erdmann-Henke, 2002)

Let $s = s_0 + ps'$ and $t = t_0 + pt'$ with $0 \leqslant s_0, t_0 \leqslant p-1$ and $s', t' \geqslant 0$.

(1) If $p = 2$, then

$$n(v^s, v^t) = \begin{cases} n(v^{s'}, v^{t'}) & \text{if } s_0 = t_0 = 1 \text{ or } s_0 = t_0 = 0 \text{ and } s' \equiv t' \mod 2, \\ 1 & \text{if } s_0 = t_0 = 0, t' + 1 = s' \not\equiv 0 \mod 2, \\ 0 & \text{otherwise.} \end{cases}$$

(2) If $p > 2$, then

$$n(v^s, v^t) = \begin{cases} n(v^{s'}, v^{t'}) & \text{if } s_0 = t_0, \\ 1 & \text{if } s_0 + t_0 = p-2, t' + 1 = s' \not\equiv 0 \mod p, \\ 0 & \text{otherwise.} \end{cases}$$

Example 1

Let $p = 2$. We consider the quiver of the basic algebra of $S(2, 10)$. By Theorem 2.2, we have

$$\begin{array}{ccccccc}
 & & & & (6,4) & \rightleftarrows & (10) \\
 & & & & \uparrow\downarrow & & \uparrow\downarrow \\
(8,2) & \rightleftarrows & (7,3) & \rightleftarrows & (5^2) & \rightleftarrows & (9,1)
\end{array}.$$

Example 2

Let $p = 2$. The quiver of the basic algebra of $S(2, 11)$ is

$$(10,1) \rightleftarrows (6,5) \rightleftarrows (8,3) \qquad (11) \rightleftarrows (7,4) \qquad (9,2)\ .$$

(Cont'd) To find the relations for the basic algebra of $S(2, 11)$, we need to find the explicit structures of Young modules which are uniquely determined by

- the decomposition matrix $[S^\lambda : D^\mu]$, see [James, 1978],
- the filtration multiplicities $[Y^\lambda : S^\mu]$, or equivalently, the character ch Y^λ, see [Henke, 1999].

Let B_1 be the principal block of $\mathbb{F}G_{11}$ and B_2 the block of $\mathbb{F}G_{11}$ labeled by 2-core $(2, 1)$, the parts of the decomposition matrix $[S^\lambda : D^\mu]$ for the partitions in B_1 and B_2 with at most two parts are

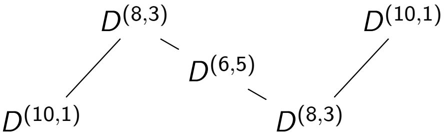
$$B_1 : \begin{matrix} (11) \\ (9,2) \\ (7,4) \end{matrix} \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 1 & 0 & 1 \end{pmatrix}, \; B_2 : \begin{matrix} (10,1) \\ (8,3) \\ (6,5) \end{matrix} \begin{pmatrix} 1 & & \\ 1 & 1 & \\ 0 & 1 & 1 \end{pmatrix}.$$

We next explain how to determine S_{B_2}.

(Cont'd) By [Henke, 1999], we have

$$\begin{aligned} \text{ch } Y^{(10,1)} &= \chi^{(10,1)}, \\ \text{ch } Y^{(8,3)} &= \chi^{(10,1)} + \chi^{(8,3)}, \\ \text{ch } Y^{(6,5)} &= \chi^{(10,1)} + \chi^{(8,3)} + \chi^{(6,5)}. \end{aligned}$$

Then, we have

- $Y^{(10,1)} = D^{(10,1)}$, $Y^{(8,3)} = \begin{smallmatrix} D^{(10,1)} \\ D^{(8,3)} \\ D^{(10,1)} \end{smallmatrix}$,
- $Y^{(6,5)} =$  .

(Cont'd) Then, $S_{B_2} = \mathrm{End}_{\mathbb{F}G_{11}}(Y^{(10,1)} \oplus Y^{(8,3)} \oplus Y^{(6,5)})$ is isomorphic to $\mathbb{F}Q/I$ with

$$Q: \ (10,1) \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} (6,5) \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} (8,3)$$

and

$$I: \langle \alpha_1\beta_1, \beta_2\alpha_2, \alpha_1\alpha_2\beta_2, \alpha_2\beta_2\beta_1 \rangle.$$

Similarly, $S_{B_1} = \mathbb{F}\left((11) \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} (7,4) \qquad (9,2) \right) / \langle \alpha_1\beta_1 \rangle.$

We point out the following facts.

- We may compute the basic algebra of $S(2, r)$ by hand.
- We may compute the basic algebra of $S(n, r)$ with $n \geqslant 3$ by a computer. For example, we refer to Carlson and Matthews's program:

http://alpha.math.uga.edu/ jfc/schur.html.

The complete classification of representation type

The representation type of Schur algebras is completely determined by [Erdmann, 1993], [Xi, 1993], [Doty-Nakano, 1998] and [Doty-Erdmann-Martin-Nakano, 1999].

Proposition 2.3

Let char $\mathbb{F} = 0$ or p. Then, $S(n, r)$ is

- semi-simple if and only if char $\mathbb{F} = 0$ or $p > r$ or $p = 2, n = 2, r = 3$;
- representation-finite if and only if $p = 2, n = 2, r = 5, 7$ or $p \geqslant 2, n = 2, r < p^2$ or $p \geqslant 2, n \geqslant 3, r < 2p$;
- infinite-tame if and only if $p = 2, n = 2, r = 4, 9, 11$ or $p = 3, n = 2, r = 9, 10, 11$ or $p = 3, n = 3, r = 7, 8$.

Otherwise, $S(n, r)$ is wild.

Tame Schur algebras

Representation-finite blocks of Schur algebras

Proposition 3.1 (Erdmann, 1993; Donkin-Reiten, 1994)

Let A be a representation-finite block of Schur algebras. Then, it is Morita equivalent to $\mathcal{A}_m := \mathbb{F}Q/I$ for some $m \in \mathbb{N}$, where

$$Q:\ 1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \cdots \underset{\beta_{m-2}}{\overset{\alpha_{m-2}}{\rightleftarrows}} m-1 \underset{\beta_{m-1}}{\overset{\alpha_{m-1}}{\rightleftarrows}} m\ ,$$

$$I:\langle \alpha_1\beta_1, \alpha_i\alpha_{i+1}, \beta_{i+1}\beta_i, \beta_i\alpha_i - \alpha_{i+1}\beta_{i+1} \mid 1 \leqslant i \leqslant m-2\rangle.$$

Theorem 3.2 (W, 2020)

We have $\#\text{s}\tau\text{-tilt}\ \mathcal{A}_m = \binom{2m}{m}$.

Sketch of the proof: We remark that $\mathcal{A}_m$ is a quotient of a Brauer line algebra modulo the two-sided ideal generated by the central element $\alpha_1\beta_1$. Then, the number follows from [Aoki, 2018].

Tame blocks of tame Schur algebras

Proposition 3.3 (Doty-Erdmann-Martin-Nakano, 1999)

Let A be an infinite-tame block of tame Schur algebras. Then, it is Morita equivalent to one of $\mathcal{D}_3$, $\mathcal{D}_4$, $\mathcal{R}_4$ and $\mathcal{H}_4$, where

- $\mathcal{D}_3 := \mathbb{F}Q/I$ is the special biserial algebra given by

$$Q: \circ \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \circ \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \circ \text{ and } I: \langle \alpha_1\beta_1, \beta_2\alpha_2, \alpha_1\alpha_2\beta_2, \alpha_2\beta_2\beta_1 \rangle.$$

- $\mathcal{D}_4 := \mathbb{F}Q/I$ is the bound quiver algebra given by

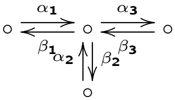
$$Q: \begin{array}{ccccc} \circ & \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} & \circ & \underset{\alpha_3}{\overset{\beta_3}{\rightleftarrows}} & \circ \\ & & {\scriptstyle \alpha_2}\uparrow\downarrow{\scriptstyle \beta_2} & & \\ & & \circ & & \end{array} \text{ and } I: \left\langle \begin{array}{c} \alpha_1\beta_1, \alpha_2\beta_2, \alpha_3\beta_1, \alpha_3\beta_2, \alpha_1\beta_3, \alpha_2\beta_3, \\ \alpha_1\beta_2\alpha_2, \beta_2\alpha_2\beta_1, \beta_2\alpha_2 - \beta_3\alpha_3 \end{array} \right\rangle.$$

- $\mathcal{R}_4 := \mathbb{F}Q/I$ is the bound quiver algebra given by

$$Q: \circ \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \circ \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \circ \underset{\beta_3}{\overset{\alpha_3}{\rightleftarrows}} \circ \text{ and } I: \left\langle \begin{array}{c} \alpha_1\beta_1, \alpha_1\alpha_2, \beta_2\beta_1, \\ \alpha_2\beta_2 - \beta_1\alpha_1, \alpha_3\beta_3 - \beta_2\alpha_2 \end{array} \right\rangle.$$

(Cont'd)

- $\mathcal{H}_4 := \mathbb{F}Q/I$ is the bound quiver algebra given by

$Q:$ 

and $I: \left\langle \begin{matrix} \alpha_1\beta_1, \alpha_1\beta_2, \alpha_2\beta_1, \alpha_2\beta_2, \alpha_1\alpha_3, \\ \beta_3\beta_1, \alpha_3\beta_3 - \beta_1\alpha_1 - \beta_2\alpha_2 \end{matrix} \right\rangle$.

Lemma 3.4 (W, 2020)

The algebras $\mathcal{D}_3$, $\mathcal{D}_4$, $\mathcal{R}_4$ and $\mathcal{H}_4$ are τ-tilting finite. Moreover,

A	$\mathcal{D}_3$	$\mathcal{D}_4$	$\mathcal{R}_4$	$\mathcal{H}_4$
$\#\mathrm{s}\tau\text{-tilt}\, A$	28	114	88	96

.

Sketch of the proof: We show this lemma by direct computation with some reduction steps.

Theorem 3.5 (W, 2020)

If the Schur algebra $S(n, r)$ is tame, then it is τ-tilting finite.

Sketch of the proof: We have showed that $\mathcal{A}_m$, $\mathcal{D}_3$, $\mathcal{D}_4$, $\mathcal{R}_4$ and $\mathcal{H}_4$ are τ-tilting finite. It suffices to make clear that these are all the blocks of tame Schur algebras. For example, let $p = 3$, then

- the basic algebra of $S(2, 9)$ is isomorphic to $\mathcal{D}_4 \oplus \mathbb{F}$;
- the basic algebra of $S(2, 10)$ is isomorphic to $\mathcal{D}_4 \oplus \mathbb{F} \oplus \mathbb{F}$;
- the basic algebra of $S(2, 11)$ is isomorphic to $\mathcal{D}_4 \oplus \mathcal{A}_2$;
- the basic algebra of $S(3, 7)$ is isomorphic to $\mathcal{R}_4 \oplus \mathcal{A}_2 \oplus \mathcal{A}_2$;
- the basic algebra of $S(3, 8)$ is isomorphic to $\mathcal{R}_4 \oplus \mathcal{H}_4 \oplus \mathcal{A}_2$.

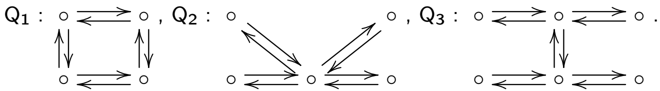
Wild Schur algebras

Strategy on τ-tilting infinite Schur algebras

Let $A := \mathbb{F}Q/I$ be an algebra. We call Q a τ-tilting infinite quiver if $A/\mathrm{rad}^2 A$ is τ-tilting infinite. For example, the Kronecker quiver $Q: \circ \rightrightarrows \circ$ is a τ-tilting infinite quiver.

Lemma 4.1 (W, 2020)

The following $\mathrm{Q}_1, \mathrm{Q}_2$ and Q_3 are τ-tilting infinite quivers.



<u>Sketch of the proof:</u> We remark that representation-infinite path algebras are τ-tilting infinite.

Reduction theorems on $S(n, r)$

Lemma 4.2 (W, 2020)

If $S(n, r)$ is τ-tilting infinite, then so is $S(N, r)$, for any $N > n$.

Sketch of the proof: We show that $S(n, r)$ is an idempotent truncation of $S(N, r)$.

Lemma 4.3 (W, 2020)

If $S(n, r)$ is τ-tilting infinite, then so is $S(n, n + r)$.

Sketch of the proof: It is shown by [Erdmann, 1993] that $S(n, r)$ is a quotient of $S(n, n + r)$.

We consider wild Schur algebras except for the following cases.

$$(\star)\quad \begin{cases} p=2, n=2, r=8,17,19; \\ p=2, n=3, r=4; \\ p=2, n\geqslant 5, r=5; \\ p\geqslant 5, n=2, p^2\leqslant r\leqslant p^2+p-1. \end{cases}$$

The characteristic $p = 2$

Blue: τ-tilting finite **Red**: τ-tilting infinite

r	1	2	3	4	5	6	7	8	9	10	11	12
$S(2,r)$	S	F	S	T	F	W	F	W	T	W	T	W

r	13	14	15	16	17	18	19	20	21	22	23	...
$S(2,r)$	W	W	W	W	W	W	W	W	W	W	W	...

Sketch of the proof: We show the following.

- the basic algebra $\overline{S(2,6)}$ of $S(2,6)$ is τ-tilting finite;
- the basic algebra of $S(2,10)$ is τ-tilting infinite ($\Leftarrow$ Q$_1$);
- the basic algebra of $S(2,13)$ is $\overline{S(2,6)} \oplus \mathcal{A}_2 \oplus \mathbb{F}$;
- the basic algebra of $S(2,15)$ is $\overline{S(2,6)} \oplus \mathcal{A}_2 \oplus \mathbb{F} \oplus \mathbb{F}$;
- the basic algebra of $S(2,21)$ is τ-tilting infinite ($\Leftarrow$ Q$_1$).

The characteristic $p = 2$

$n \backslash r$	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
3	S	F	F	W	W	W	W	W	W	W	W	W	W	$\cdots$
4	S	F	F	W	W	W	W	W	W	W	W	W	W	$\cdots$
5	S	F	F	W	W	W	W	W	W	W	W	W	W	$\cdots$
6	S	F	F	W	W	W	W	W	W	W	W	W	W	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Sketch of the proof:

- the basic algebra of $S(3,6)$ is τ-tilting infinite ($\Leftarrow$ Q_1);
- the basic algebra of $S(3,7)$ is τ-tilting infinite ($\Leftarrow$ Q_1);
- the basic algebra of $S(3,8)$ is τ-tilting infinite ($\Leftarrow$ Q_2);
- the basic algebra of $S(4,4)$ is τ-tilting infinite ($\Leftarrow$ Q_1);
- the basic algebra of $S(4,5)$ is τ-tilting finite.

The characteristic $p = 3$

n \ r	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
2	S	S	F	F	F	F	F	F	T	T	T	W	W	$\cdots$
3	S	S	F	F	F	W	T	T	W	W	W	W	W	$\cdots$
4	S	S	F	F	F	W	W	W	W	W	W	W	W	$\cdots$
5	S	S	F	F	F	W	W	W	W	W	W	W	W	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Sketch of the proof:

- the basic algebras of $S(2,12)$ and $S(2,13)$ are τ-tilting infinite ($\Leftarrow Q_1$);
- the basic algebra of $S(3,6)$ is τ-tilting infinite ($\Leftarrow Q_1$);
- the basic algebras of $S(3,10)$ and $S(3,11)$ are τ-tilting infinite ($\Leftarrow Q_3$);
- the basic algebra of $S(4,7)$ is τ-tilting infinite ($\Leftarrow Q_2$);
- the basic algebra of $S(4,8)$ is τ-tilting infinite ($\Leftarrow Q_1$).

The characteristic $p \geqslant 5$

n \ r	$1 \sim p-1$	$p \sim 2p-1$	$2p \sim p^2-1$	$p^2 \sim p^2+p-1$	$p^2+p \sim \infty$
2	S	F	F	W	W
3	S	F	W	W	W
4	S	F	W	W	W
5	S	F	W	W	W
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Sketch of the proof:

- the basic algebras of $S(2, p^2+p)$ and $S(2, p^2+p+1)$ are τ-tilting infinite ($\Leftarrow \mathrm{Q}_1$);
- the basic algebras of $S(3, 2p)$, $S(3, 2p+1)$ and $S(3, 2p+2)$ are τ-tilting infinite ($\Leftarrow \mathrm{Q}_3$).

Theorem 4.4 (W, 2020)

Let $p \geqslant 5$ and except for the cases in $(\star)$, the Schur algebra $S(n,r)$ satisfies the condition:

$$\tau\text{-titling finite} \Leftrightarrow \text{representation-finite}$$

Theorem 4.5 (W, 2020)

Let $p = 3$, the Schur algebra $S(n,r)$ satisfies the condition:

$$\tau\text{-titling infinite} \Leftrightarrow \text{wild}.$$

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Thank you very much for your attention !