τ-tilting finiteness of two-point algebras

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Outline

Auslander-Reiten translation

Let Λ be a finite dimensional basic algebra over an algebraically closed field K. For a Λ-module M with a minimal projective presentation

$$P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0,$$

we have

$$\mathrm{Tr} M := \operatorname{coker} \mathrm{Hom}_\Lambda(d_1, \Lambda).$$

The Auslander-Reiten translation is defined by

$$\tau M := D\mathrm{Tr} M,$$

where $D = \mathrm{Hom}_K(-, K)$.

Introduction

τ-tilting theory was introduced by Adachi-Iyama-Reiten as a generalization of the classical tilting theory via mutations.

They introduced support τ-tilting modules for finite dimensional algebras and constructed the (left or right) mutation of support τ-tilting modules, which has the following nice properties:

- Mutation (left or right) is always possible.
- There is a partial order on the set of (isomorphism classes of) basic support τ-tilting modules such that its Hasse quiver realizes the support τ-tilting mutations.

Motivations

We call an algebra Λ τ-tilting finite if there are only finitely many (isomorphism classes of) basic support τ-tilting Λ-modules.

Recently, many scholars have studied the support τ-tilting modules of various algebras. For example,

- Gentle algebras (Plamondon, 2018).
- Tilted and cluster-tilted algebras (Zito, 2019).
- Algebras with radical square zero (Adachi, 2016).
- Brauer graph algebras (Adachi-Aihara-Chan, 2018).
- Preprojective algebras of Dynkin type (Aihara-Mizuno, 2016, Mizuno, 2014).

Note that local algebras are τ-tilting finite, and any idempotent truncation of a τ-tilting finite algebra is also τ-tilting finite. Thus, considering τ-tilting finiteness of two-point algebras is fundamental for determining τ-tilting finiteness of general algebras.

Furthermore, we have to consider quivers that contain loops. Because Gabriel quivers have loops in general and most of the previous cases are dealing with algebras which do not have loops.

Basics of τ-tilting theory

We denote by $|M|$ the number of (isomorphism classes of) indecomposable direct summands of M.

Definition 1.1 (Adachi-Iyama-Reiten, 2014)

Let M be a right Λ-module and $P \in \operatorname{proj} \Lambda$.

(1) M is called τ-rigid if $\operatorname{Hom}_\Lambda(M, \tau M) = 0$.

(2) M is called τ-tilting if M is τ-rigid and $|M| = |\Lambda|$.

(3) M is called support τ-tilting if M is a τ-tilting $(\Lambda/\langle e \rangle)$-module, where e is an idempotent of Λ.

(4) (M, P) is called a support τ-tilting pair if M is τ-rigid, $\operatorname{Hom}_\Lambda(P, M) = 0$ and $|M| + |P| = |\Lambda|$.

We denote by τ-rigid Λ (respectively, sτ-tilt Λ) the set of (isomorphism classes of) indecomposable τ-rigid (respectively, basic support τ-tilting) Λ-modules.

Example

Let $\Lambda = K(1 \underset{b}{\overset{a}{\rightleftarrows}} 2)/\{ab = ba = 0\}$ and we denote

$$S_1 = 1, S_2 = 2, P_1 = {}^{1}_{2}, P_2 = {}^{2}_{1}.$$

Then,

$$\tau(S_1) = S_2, \tau(S_2) = S_1, \tau(P_1) = 0, \tau(P_2) = 0.$$

Thus,

- S_1, S_2, P_1 and P_2 are τ-rigid modules.
- $P_1 \oplus P_2, P_1 \oplus S_1$ and $S_2 \oplus P_2$ are τ-tilting modules.
- S_1 and S_2 are also support τ-tilting modules.
- (S_1, P_2) and (S_2, P_1) are support τ-tilting pairs.

Mutation

We denote by add(M) (respectively, Fac(M)) the full subcategory whose objects are direct summands (respectively, factor modules) of finite direct sums of copies of M.

Definition 1.2 (Adachi-Iyama-Reiten, 2014)

Let $T = M \oplus N$ be a basic τ-tilting module, where $M \notin \mathrm{Fac}(N)$ is an indecomposable summand. We take an exact sequence with a minimal left add(N)-approximation π:

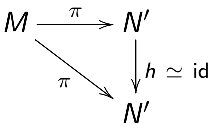
$$M \xrightarrow{\pi} N' \longrightarrow U \longrightarrow 0,$$

we call $\mu_M^-(T) := U \oplus N$ the left mutation of T with respect to M.

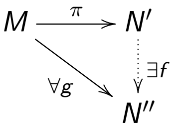
Remark

π is called a minimal left add(N)-approximation if $N' \in$ add(N) and it satisfies the following conditions:

(i) every $h : N' \to N'$ that satisfies $h \circ \pi = \pi$ is an automorphism.



(ii) for any $N'' \in$ add(N) and $g : M \to N''$, there exists $f : N' \to N''$ such that $f \circ \pi = g$.



Example

Let $\Lambda = K(\ 1 \underset{b}{\overset{a}{\rightleftarrows}} 2\ )/\{ab = ba = 0\}$, then $P_1 \oplus P_2$ is a τ-tilting module. We consider the left mutation with respect to P_2,

$$P_2 \xrightarrow{\pi} P_1 \longrightarrow \mathrm{coker}\pi \longrightarrow 0,$$

where $\pi : \begin{smallmatrix} e_2 \\ b \end{smallmatrix} \xrightarrow{a\cdot} \begin{smallmatrix} e_1 \\ a \end{smallmatrix}$ is a minimal left $\mathrm{add}(P_1)$-approximation,

$$\mathrm{coker}\pi = S_1 \text{ and } \mu^{-}_{P_2}(\Lambda) = P_1 \oplus S_1.$$

In fact, we have the following mutation quiver of sτ-tilt Λ.

$$\begin{array}{ccccc}
P_1 \oplus P_2 & \longrightarrow & P_1 \oplus S_1 & \longrightarrow & S_1 \\
\downarrow & & & & \downarrow \\
S_2 \oplus P_2 & \longrightarrow & S_2 & \longrightarrow & 0
\end{array}$$

Proposition 1.3 (Adachi-Iyama-Reiten, 2014)

If the mutation quiver $\mathcal{Q}(\mathsf{s}\tau\text{-tilt}\ \Lambda)$ contains a finite connected component, then it exhausts all support τ-tilting modules.

Proposition 1.4 (Adachi-Iyama-Reiten, 2014)

Any τ-rigid module is a direct summand of some τ-tilting modules.

Poset structure

Definition 1.5 (Adachi-Iyama-Reiten, 2014)

For $M, N \in \text{s}\tau\text{-tilt}\,\Lambda$, we say $M \geqslant N$ if $\text{Fac}(N) \subseteq \text{Fac}(M)$.

Example

Let Λ be the algebra given before. The Hasse quiver of $\text{s}\tau\text{-tilt}\,\Lambda$ is

$$\begin{array}{ccccc}
P_1 \oplus P_2 & \xrightarrow{>} & P_1 \oplus S_1 & \xrightarrow{>} & S_1 \\
\downarrow{\scriptstyle >} & & & & \downarrow{\scriptstyle >} \\
S_2 \oplus P_2 & \xrightarrow{>} & S_2 & \xrightarrow{>} & 0
\end{array}$$

Proposition 1.6 (Adachi-Iyama-Reiten, 2014)

The mutation quiver $\mathcal{Q}(\text{s}\tau\text{-tilt}\,\Lambda)$ and the Hasse quiver $\mathcal{H}(\text{s}\tau\text{-tilt}\,\Lambda)$ coincide.

Basics of silting theory

Let $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\ \Lambda)$ be the homotopy category of bounded complexes of projective Λ-modules. For $T \in \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\ \Lambda)$, let thick T be the smallest full triangulated subcategory containing T, which is closed under isomorphic classes, cones, $[\pm 1]$ and direct summands.

Definition 1.7 (Aihara-Iyama, 2012)

If thick $T = \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\ \Lambda)$ and

$$\mathrm{Hom}_{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\ \Lambda)}(T, T[i]) = 0$$

for any $i > 0$, then we call T a silting complex.

A complex T in $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\ \Lambda)$ is called two-term if it is concentrated in degree 0 and -1. We denote by 2-silt Λ the set of (isomorphism classes of) basic two-term silting complexes in $K^{b}(\mathsf{proj}\ \Lambda)$.

Proposition 1.8 (Adachi-Iyama-Reiten, 2014)

There exists a poset isomorphism between sτ-tilt Λ and 2-silt Λ. More precisely, the bijection is given by

$$\begin{array}{ccc} \mathrm{s}\tau\text{-tilt}\,\Lambda & \longmapsto & 2\text{-silt}\,\Lambda \\ M & \longmapsto & (P_1 \oplus P \xrightarrow{[f,0]} P_0) \end{array}$$

where (M, P) is a support τ-tilting pair and $f : P_1 \to P_0$ is a minimal projective presentation of M.

By the previous proposition, we have the poset structure on 2-silt Λ in the following example.

Example

Let $\Lambda = K(\ 1 \underset{b}{\overset{a}{\rightleftarrows}} 2\ )/\{ab = ba = 0\}$. The mutation quiver $\mathcal{Q}(\text{2-silt } \Lambda)$ and the Hasse quiver $\mathcal{H}(\text{2-silt } \Lambda)$ also coincide.

$$
\begin{array}{ccccc}
\begin{bmatrix} 0 \longrightarrow P_1 \\ \oplus \\ 0 \longrightarrow P_2 \end{bmatrix} & \longrightarrow & \begin{bmatrix} 0 \longrightarrow P_1 \\ \oplus \\ P_2 \overset{a}{\longrightarrow} P_1 \end{bmatrix} & \longrightarrow & \begin{bmatrix} P_2 \longrightarrow 0 \\ \oplus \\ P_2 \overset{a}{\longrightarrow} P_1 \end{bmatrix} \\
\downarrow & & & & \downarrow \\
\begin{bmatrix} P_1 \overset{b}{\longrightarrow} P_2 \\ \oplus \\ 0 \longrightarrow P_2 \end{bmatrix} & \longrightarrow & \begin{bmatrix} P_1 \overset{b}{\longrightarrow} P_2 \\ \oplus \\ P_1 \longrightarrow 0 \end{bmatrix} & \longrightarrow & \begin{bmatrix} P_1 \longrightarrow 0 \\ \oplus \\ P_2 \longrightarrow 0 \end{bmatrix}
\end{array}
$$

Reduction theorems

Recall that M is called a *brick* if $\mathrm{End}_\Lambda(M) = K$ and let brick Λ be the set of (isomorphism classes of) indecomposable bricks in mod Λ. We call Λ_2 a *factor algebra* of Λ_1 if there is a surjective K-algebra homomorphism $\phi : \Lambda_1 \longrightarrow \Lambda_2$.

Lemma 1.9 (Demonet-Iyama-Jasso, 2015)

There is a bijection between τ-rigid Λ and brick Λ, which is given by

$$M \to M/\mathrm{rad}_B M,$$

where M is a τ-rigid module and $B = \mathrm{End}_\Lambda M$.

Corollary 1.10 (W, 2019)

Let Λ_2 is a factor algebra of Λ_1. If Λ_2 is τ-tilting infinite, so is Λ_1.

Lemma 1.11 (Adachi-Iyama-Reiten, 2014)

There exists a poset isomorphism between sτ-tilt Λ and sτ-tilt Λ^{op}.

Lemma 1.12 (Eisele-Janssens-Raedschelders, 2018)

Let I be a two-sided ideal generated by elements which are contained in the center and the radical, then there exists a poset isomorphism between sτ-tilt Λ and sτ-tilt (Λ/I).

Lemma 1.13 (Plamondon, 2018)

Any gentle algebra satisfies

$$\tau\text{-tilting finite} \Leftrightarrow \text{representation-finite}.$$

Main Results

Definition 2.1

Let Λ be a τ-tilting finite algebra. The Hasse quiver $\mathcal{H}(\mathsf{s}\tau\text{-tilt}\,\Lambda)$ is of type $\mathcal{H}_{m,n}$ if it is of the form

$$\begin{array}{ccccccccccc} & \nearrow & \triangle_1 & \longrightarrow & \triangle_2 & \longrightarrow & \cdots & \longrightarrow & \triangle_m & \searrow & \\ \Lambda & & & & & & & & & & 0. \\ & \searrow & \square_1 & \longrightarrow & \square_2 & \longrightarrow & \cdots & \longrightarrow & \square_n & \nearrow & \end{array}$$

Note that $\mathcal{H}_{m,n} \simeq \mathcal{H}_{n,m}$.

Theorem 2.2 (W, 2019)

Let Γ_i be an algebra from the Table Γ.

(1) For $i = 1, 3, 4, 5, 6, 7$, Γ_i are minimal τ-tilting infinite.

(2) For the remaining $i = 2, 8, 9, \ldots, 20$, Γ_i are τ-tilting finite. Moreover, the number of support τ-tilting modules of Γ_i and the type of $\mathcal{H}(\text{s}\tau\text{-tilt}\,\Gamma_i)$ are as follows.

Γ_i	Γ_2	Γ_8	Γ_9	Γ_{10}	Γ_{11}	Γ_{12}	Γ_{13}
$\#\text{s}\tau\text{-tilt}\,\Gamma_i$	8	7	9	8	12	9	8
type	$\mathcal{H}_{1,5}$	$\mathcal{H}_{2,3}$	$\mathcal{H}_{2,5}$	$\mathcal{H}_{3,3}$	$\mathcal{H}_{5,5}$	$\mathcal{H}_{2,5}$	$\mathcal{H}_{3,3}$

Γ_i	Γ_{14}	Γ_{15}	Γ_{16}	Γ_{17}	Γ_{18}	Γ_{19}	Γ_{20}
$\#\text{s}\tau\text{-tilt}\,\Gamma_i$	8	12		8		10	6
type	$\mathcal{H}_{3,3}$	$\mathcal{H}_{5,5}$		$\mathcal{H}_{3,3}$	$\mathcal{H}_{2,4}$	$\mathcal{H}_{4,4}$	$\mathcal{H}_{2,2}$

Sketch of proof

(1) Γ_1: We combine the τ-rigid-brick corresponding and the fact:

$$M_\lambda = \bullet \underset{1}{\overset{\lambda}{\rightrightarrows}} \bullet \text{ is a brick.}$$

(2) Γ_4: This is a (infinite-)tame gentle algebra.

(3) Others: We compute the Hasse quiver $\mathcal{H}(\text{s}\tau\text{-tilt } \Gamma_i)$ or $\mathcal{H}(\text{2-silt } \Gamma_i)$ directly.

Theorem 2.3 (W, 2019)

Let Λ be a two-point algebras with quiver Q_Λ.

(1) If

$$Q_\Lambda = \bullet \longrightarrow \bullet \circlearrowleft,$$

then Λ is τ-tilting infinite if and only if it has Γ_3 as a factor algebra.

(2) If

$$Q_\Lambda = \circlearrowright \bullet \longrightarrow \bullet \circlearrowleft,$$

then Λ is τ-tilting infinite if and only if it has one of $\Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6$ and their opposite algebras as a factor algebra.

Sketch of proof

We give an example to explain the main method. Let

$$\Lambda = K(\ 1 \xrightarrow{a} 2 \circlearrowright b\ )/I,$$

where I is an admissible ideal such that Λ is a finite dimensional algebra. Then I is either $< b^n >$ or $< b^n, ab^m >$ with $n \geqslant m$. Note that the latter can be reduced to the former by Lemma 1.12.

Application 1

Representation type of two-point algebras has already completely determined for many years.

Proposition 3.1 (Han, 2002)

Let A be a two-point algebra. Up to isomorphism and duality, A is representation-finite or (infinite-)tame if and only if A degenerates to a factor algebra of an algebra from Table T, and A is wild if and only if A has a wild algebra from Table W as a factor algebra.

Theorem 3.2 (W, 2019)

Let T_i and W_i be algebras from Table T and Table W.

(1) $T_1, T_3, T_{17}, W_1, W_2, W_3$ and W_5 are τ-tilting infinite.

(2) Others are τ-tilting finite. Moreover, $\#\mathrm{s}\tau\text{-tilt}\ T_i$, $\#\mathrm{s}\tau\text{-tilt}\ W_i$, the type of $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\ T_i)$ and the type of $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\ W_i)$ are as follows.

<table>
<tr><td>T_2</td><td>T_4</td><td>T_5</td><td>T_6</td><td>T_7</td><td>T_8</td><td>T_9</td><td>T_{10}</td><td>T_{11}</td></tr>
<tr><td colspan="2">6</td><td>5</td><td>6</td><td colspan="2">5</td><td>8</td><td>12</td><td>8</td></tr>
<tr><td colspan="2">$\mathcal{H}_{1,3}$</td><td>$\mathcal{H}_{1,2}$</td><td>$\mathcal{H}_{1,3}$</td><td colspan="2">$\mathcal{H}_{1,2}$</td><td>$\mathcal{H}_{3,3}$</td><td>$\mathcal{H}_{5,5}$</td><td>$\mathcal{H}_{3,3}$</td></tr>
</table>

T_{12}	T_{13}	T_{14}	T_{15}	T_{16}	T_{18}	T_{19}	T_{20}	T_{21}
7	6	8	7	9	8	6	7	6
$\mathcal{H}_{2,3}$	$\mathcal{H}_{2,2}$	$\mathcal{H}_{3,3}$	$\mathcal{H}_{2,3}$	$\mathcal{H}_{2,5}$	$\mathcal{H}_{3,3}$	$\mathcal{H}_{2,2}$	$\mathcal{H}_{2,3}$	$\mathcal{H}_{2,2}$

<table>
<tr><td>W_4</td><td>W_6</td><td>W_7</td><td>W_8</td><td>W_9</td><td>W_{10}</td><td>W_{11}</td><td>W_{12}</td><td>W_{13}</td><td>W_{14}</td></tr>
<tr><td>5</td><td>6</td><td colspan="2">8</td><td colspan="2">6</td><td>7</td><td colspan="2">5</td><td>10</td></tr>
<tr><td>$\mathcal{H}_{1,2}$</td><td>$\mathcal{H}_{1,3}$</td><td colspan="2">$\mathcal{H}_{1,5}$</td><td colspan="2">$\mathcal{H}_{1,3}$</td><td>$\mathcal{H}_{1,4}$</td><td colspan="2">$\mathcal{H}_{1,2}$</td><td>$\mathcal{H}_{3,5}$</td></tr>
</table>

<table>
<tr><td>W_{15}</td><td>W_{16}</td><td>W_{17}</td><td>W_{18}</td><td>W_{19}</td><td>W_{20}</td><td>W_{21}</td><td>W_{22}</td><td>W_{23}</td><td>W_{24}</td></tr>
<tr><td>9</td><td>8</td><td>9</td><td>8</td><td colspan="4">7</td><td>8</td><td>10</td></tr>
<tr><td>$\mathcal{H}_{2,5}$</td><td>$\mathcal{H}_{3,3}$</td><td>$\mathcal{H}_{2,5}$</td><td>$\mathcal{H}_{3,3}$</td><td colspan="4">$\mathcal{H}_{2,3}$</td><td>$\mathcal{H}_{3,3}$</td><td>$\mathcal{H}_{3,5}$</td></tr>
</table>

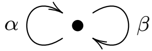
<table>
<tr><td>W_{25}</td><td>W_{26}</td><td>W_{27}</td><td>W_{28}</td><td>W_{29}</td><td>W_{30}</td><td>W_{31}</td><td>W_{32}</td><td>W_{33}</td><td>W_{34}</td></tr>
<tr><td colspan="2">7</td><td colspan="4">8</td><td colspan="4">6</td></tr>
<tr><td colspan="2">$\mathcal{H}_{2,3}$</td><td>$\mathcal{H}_{2,4}$</td><td colspan="2">$\mathcal{H}_{3,3}$</td><td>$\mathcal{H}_{2,4}$</td><td colspan="4">$\mathcal{H}_{2,2}$</td></tr>
</table>

Application 2

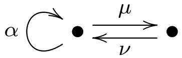
Let Λ be a block algebra of Hecke algebras of classical type over an algebraically closed field k of odd characteristic.

Proposition 3.3 (Ariki, 2018)

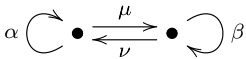
If Λ is (infinite-)tame, then it is Morita equivalent to one of the bounded quiver algebras $H_i = kQ_i/I_i$ below.

H_1: $Q_1 =$ 

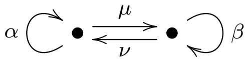
, $I_1 = < \alpha^2, \beta^2, \alpha\beta, \beta\alpha >$.

H_2: $Q_2 =$ 

, $I_2 = < \alpha\mu, \nu\alpha, \alpha^2 - (\mu\nu)^2 >$.

H_3: $Q_3 =$ 

$I_3 = < \alpha\mu, \mu\beta, \beta\nu, \nu\alpha, \alpha^2 - \mu\nu, \beta^2 - \nu\mu >$.

H_4: $Q_4 =$ 

$I_4 = < \alpha\mu, \mu\beta, \beta\nu, \nu\alpha, \alpha^2 - (\mu\nu)^2, \beta^2 - (\nu\mu)^2 >$.

Theorem 3.4 (W, 2019)

Let Λ be a (infinite-)tame block algebra, then Λ is τ-tilting finite. Moreover, we have

H_i	H_1	H_2	H_3	H_4
$\#\text{s}\tau\text{-tilt } H_i$	2	6		

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Thank you very much for your attention !