On τ-tilting finiteness of several finite-dimensional algebras

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Papers and Presentations

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- Q. Wang, τ-tilting finiteness of two-point algebras I, *Mathematical Journal of Okayama University*, **64** (2022), 117–141, http://doi.org/10.18926/mjou/62799.
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Presentations

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Abstract

In this thesis, we discuss the τ-tilting finiteness for three classes of finite-dimensional algebras over an algebraically closed field.

First, we consider the class of two-point algebras. We completely determine the τ-tilting finiteness for minimal wild two-point algebras. Based on this complete classification, we are able to determine the τ-tilting finiteness for some tame two-point algebras.

Second, we consider the class of simply connected algebras. We show that a simply connected algebra is τ-tilting finite if and only if it is representation-finite. We also show that the τ-tilting finiteness of non-sincere algebras can be reduced to that of sincere algebras. Then, by combining these new results with some previous results of other scholars, we can get a complete list of τ-tilting finite sincere simply connected algebras. Moreover, we can determine the τ-tilting finiteness for several related algebras, such as tubular algebras, hypercritical algebras, and locally hereditary algebras.

Last, we consider the class of (classical) Schur algebras. Here, we do not consider the τ-tilting finiteness blockwise even though some block algebras of Schur algebras are discussed. We determine the τ-tilting finiteness of Schur algebras except for a few small cases[1]. In particular, the τ-tilting finiteness of Schur algebras is completely determined if we consider an algebraically closed field of characteristic 3 in this thesis. This is a fundamental effort toward the τ-tilting finiteness of q-Schur algebras, infinitesimal Schur algebras and so on.

[1]These small cases are settled in another paper jointed with Toshitaka Aoki. Hence, we actually have a complete classification.

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Contents

Chapter 1

Introduction

1.1 Background

Tilting theory originated from the research of Bernšteǐn, Gelfand and Ponomarev [BGP] in 1973, in which the well-known BGP-reflection functor was introduced and was aimed to construct equivalences between module categories of finite-dimensional algebras. Then, the insights of [BGP] quickly attracted the attention of mathematicians. After decades of development, tilting theory is now considered not only as one of the main tools in the representation theory of finite-dimensional algebras, but also an essential tool in the study of many areas of mathematics. In particular, Rickard [Ric] proved that tilting theory is a necessary ingredient for constructing derived equivalence of finite-dimensional algebras over an algebraically closed field.

The crucial concept in tilting theory is the notion of tilting modules, which was first considered by Auslander, Platzeck and Reiten [APR], and was axiomatized by Brenner and Butler [BB]. One of the essential properties of tilting modules is the so-called mutations. Very roughly speaking, it can be explained as follows. Let $T = T_1 \oplus \cdots \oplus T_j \oplus \cdots \oplus T_n$ ($T_i \neq T_j$ if $i \neq j$) be an object in a subclass $\mathcal{C}$ of an additive category. If we can replace a direct summand T_j by T_j^* ($\not\simeq T_j$) via a certain procedure to get a new object $\mu_j(T) := (T/T_j) \oplus T_j^*$, so that $\mu_j(T)$ also lies in $\mathcal{C}$, then $\mu_j(T)$ is called the mutation of T with respect to T_j. From this perspective, it is known that the mutation at an indecomposable direct summand of tilting modules is not always possible.

Starting from 2014, τ-tilting theory introduced by Adachi, Iyama and Reiten [AIR] has drawn more and more attention. Here, τ is the Auslander-Reiten translation.

> **Definition.** Let A be a finite-dimensional algebra over an algebraically closed field K. A right A-module M is called *support τ-tilting* if $\mathsf{Hom}_B(M, \tau M) = 0$ and $|M| = |B|$ taking over $B := A/A(1-e)A$, namely, e is an idempotent of A such that the direct summands of $eA/(e\mathsf{rad}\ A)$ are exactly the composition factors of M. Here, $\mathsf{rad}\ A$ is the Jacobson radical of A. Moreover, a support τ-tilting module M is called *τ-tilting* if $B = A$.

A support τ-tilting module can be regarded as a generalization of tilting modules from the viewpoint of mutation, that is, the mutation at an indecomposable direct summand of support τ-tilting modules is always possible. Therefore, τ-tilting theory is an evolved form of tilting theory and has many better properties. Several mathematicians have shown that τ-tilting theory is closely related to many areas of mathematics, including categorical, combinatorial representation theory and geometric representation theory. In particular, support τ-tilting modules are in bijection with other objects, such as two-term silting complexes, functorially finite torsion classes, cluster-tilting objects, left finite semibricks and so on. We refer to [AIR] and [Asa] for more details.

Let A be a finite-dimensional algebra over an algebraically closed field K. We recall that A is called representation-finite if A has only finitely many isomorphism classes of indecomposable A-modules. Otherwise, we say that A is representation-infinite. For example, if $A := K\Delta$ is a path algebra of a finite quiver Δ, then A is representation-finite precisely when the underlying graph of Δ is a disjoint union of Dynkin diagrams of type $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7$ and $\mathbb{E}_8$. This is the so-called Gabriel's theorem due to Peter Gabriel.

Similar to the representation-finiteness, a modern notion named *τ-tilting finiteness* is introduced by Demonet, Iyama and Jasso [DIJ] in 2015.

> **Definition.** A finite-dimensional algebra A over an algebraically closed field K is called τ-tilting finite if there are only finitely many pairwise non-isomorphic basic τ-tilting A-modules. Otherwise, A is said to be τ-tilting infinite.

For the examples of τ-tilting finite algebras, one can immediately find that representation-finite algebras are τ-tilting finite. Besides, local algebras, i.e., algebras with only one simple module, are known to be τ-tilting finite. However, it is not easy to check whether a representation-infinite algebra is τ-tilting finite or not, such that the following question attracts our attention.

> **Question.** Let A be a representation-infinite algebra over an algebraically closed field K. Is there a way to check the τ-tilting finiteness of A?

It is worth mentioning that the τ-tilting finiteness for several classes of algebras has been determined, such as algebras with radical square zero [Ad1], preprojective algebras of Dynkin type [Mi], Brauer graph algebras [AAC], biserial algebras [Mo] and so on. In particular, it has been proved in some cases that τ-tilting finiteness coincides with representation-finiteness, including gentle algebras [Pl], cycle-finite algebras [MS], tilted and cluster-tilted algebras [Zi], quasi-tilted algebras [AHMW] and so on.

According to the currently known results, we have a characterization for the τ-tilting finiteness of path algebras, which is similar to Gabriel's theorem. We present this characterization here and one can find a proof in Remark 4.2.2: If $A = K\Delta$ is a path algebra of a finite quiver Δ, then A is τ-tilting finite if and only if the underlying graph of Δ is a disjoint union of Dynkin diagrams of type $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7$ and $\mathbb{E}_8$.

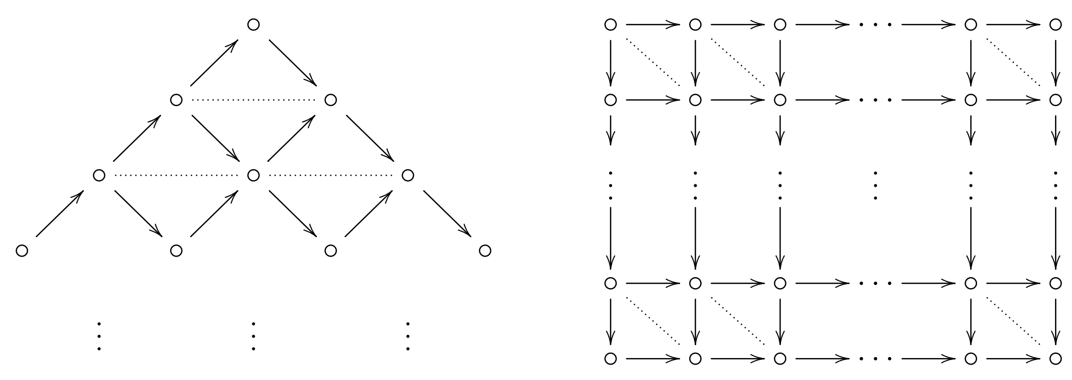
1.2 The subject of this thesis

We always assume that A is an associative unital finite-dimensional algebra over an algebraically closed field K. If A is an indecomposable algebra, then A is also assumed to be basic. Without loss of generality, such an algebra A is isomorphic to a bound quiver algebra KQ/I, where KQ is the path algebra of the Gabriel quiver $Q = Q_A$ of A and I is an admissible ideal of KQ. In fact, KQ/I gives a complete representation theoretical description of A: the number of vertices in Q is just the number of simple A-modules, the arrows and loops in Q together with the ideal I encode the structure of indecomposable projective (injective) A-modules, and many other properties of A can be read off.

We recall that the Kronecker algebra $K(\circ \rightrightarrows \circ)$ is τ-tilting infinite (see Lemma 3.1.1 for a proof) and this is actually well-known. If Q has multiple arrows, namely, there exist at least two arrows which share the source and the target, then $A \simeq KQ/I$ is τ-tilting infinite for any admissible ideal I, because A admits $K(\circ \rightrightarrows \circ)$ as a quotient algebra (see Proposition 2.3.5).

We also notice that local algebras, i.e., algebras with only one simple module, are always τ-tilting finite. Indeed, let B be a local algebra. If a B-module M satisfies $\tau M \neq 0$, then there exists a non-zero map $M \to \mathsf{soc}\, \tau M$, so that M is τ-tilting if and only if $M = B$. This motivates the study of two-point algebras, which we will consider in Chapter 3. In fact, the τ-tilting finiteness of minimal wild two-point algebras is completely determined, where A is called *minimal wild* if A is wild but any proper quotient of A is not. We find that most of minimal wild two-point algebras are τ-tilting finite.

In the process of studying two-point algebras, the main variables depend on loops and oriented cycles. Thus, we want to see what happens if $A \simeq KQ/I$ does not have multiple arrows, loops and oriented cycles, but has many vertices. As the first example of such algebras, we may consider triangle quiver and rectangle quiver as follows.



Triangle quiver **Rectangle quiver**

In these examples, the algebra $A \simeq KQ/I$ is presented by a triangle or rectangle quiver Q and an two-sided ideal I generated by all possible commutativity relations (depicted by dotted lines). Here, a commutativity relation stands for the equality $w_1 = w_2$ of two different paths w_1 and w_2 having the same source and target.

This motivates the study of simply connected algebras, which is a rather large class of algebras and contains triangle and rectangle quivers as special cases. We deal with simply connected algebras in Chapter 4. We show that a simply connected algebra is τ-tilting finite if and only if it is representation-finite. Moreover, we recall that an algebra A is called *sincere* if there exists an indecomposable A-module M such that all simple A-modules appear in M as composition factors. Otherwise, A is called non-sincere. Regarding the algebras with many vertices, we show that the τ-tilting finiteness of non-sincere algebras can be reduced to that of sincere algebras. Based on this, we get a complete list of τ-tilting finite sincere simply connected algebras.

We now explain the structure of this thesis. The main results of each chapter are also mentioned here. In Chapter 2, we review the background of the representation theory of finite-dimensional algebras and the basic definitions in τ-tilting theory. Then, we describe in detail how τ-tilting theory and silting theory are related to each other. Several reduction theorems on τ-tilting finiteness of algebras are also presented in this chapter.

In Chapter 3, we completely determine the τ-tilting finiteness of minimal wild two-point algebras. Also, by using this result, we can determine the τ-tilting finiteness for several other two-point algebras. The main result in this chapter is the following.

Main Theorem 1 (Theorem 3.2.3)**.** Let W_i be a minimal wild two-point algebra from Table W (see Appendix A.1). Then,

(1) W_1, W_2, W_3 and W_5 are τ-tilting infinite.

(2) W_4 and $W_6 \sim W_{34}$ are τ-tilting finite.

In Chapter 4, we prove that τ-tilting finiteness coincides with representation-finiteness in the class of simply connected algebras. We also determine the τ-tilting finiteness for several related algebras, such as tubular algebras, hypercritical algebras and locally hereditary algebras. The main results in this chapter are presented as follows.

Main Theorem 2 (Theorem 4.2.3)**.** Let A be a simply connected algebra. Then, A is τ-tilting finite if and only if it is representation-finite.

Main Theorem 3 (Theorem 4.2.8)**.** Let $\{e_1, e_2, \ldots, e_n\}$ be a complete set of pairwise orthogonal primitive idempotents of A. If A is non-sincere, then A is τ-tilting finite if and only if A/Ae_iA is τ-tilting finite for any $1 \leqslant i \leqslant n$.

In Chapter 5, we determine the τ-tilting finiteness of Schur algebras except for the cases in $(\star)$. These exceptional cases are settled in a joint work with Toshitaka Aoki, see [AW], so that the classification is actually complete. For the convenience of readers, we present the related results in the end of Chapter 5.

$$(\star) \quad \begin{cases} p = 2, n = 2, r = 8, 17, 19; \\ p = 2, n = 3, r = 4; \\ p = 2, n \geqslant 5, r = 5; \\ p \geqslant 5, n = 2, p^2 \leqslant r \leqslant p^2 + p - 1. \end{cases}$$

Let n, r be two positive integers and V an n-dimensional vector space over an algebraically closed field $\mathbb{F}$ of characteristic p. We denote by $V^{\otimes r}$ the r-fold tensor product $V \otimes_{\mathbb{F}} V \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} V$. Then, the symmetric group G_r has a natural action (by permutation) on $V^{\otimes r}$ which makes it a module over the group algebra $\mathbb{F}G_r$ of G_r. Then, the endomorphism ring $\mathsf{End}_{\mathbb{F}G_r}(V^{\otimes r})$ is called the Schur algebra and we denote it by $S(n, r)$. The main result in Chapter 5 is the following.

Main Theorem 4 (Theorem 5.2.4, Table 5.1, Table 5.2, Table 5.3)**.** Except for the cases in $(\star)$, the Schur algebra $S(n, r)$ is τ-tilting finite if and only if one of the following holds.

(1) $p = 0$ or $p > r$;
(2) $p = 2$, $n = 2$, $2 \leqslant r \leqslant 7$ or $r = 9, 11, 13, 15$;
(3) $p = 2$, $n = 3$ or 4, $r = 2, 3, 5$;
(4) $p = 2$, $n \geqslant 5$, $r = 2, 3$;
(5) $p = 3$, $n = 2$, $3 \leqslant r \leqslant 11$;
(6) $p = 3$, $n = 3$, $r = 3, 4, 5, 7, 8$;
(7) $p = 3$, $n \geqslant 4$, $r = 3, 4, 5$;
(8) $p \geqslant 5$, $n = 2$, $p \leqslant r \leqslant p^2 - 1$;
(9) $p \geqslant 5$, $n \geqslant 3$, $p \leqslant r \leqslant 2p - 1$.

In order to prove the above result, we have to check the τ-tilting finiteness for some block algebras of $S(n, r)$. Then, we get the number of pairwise non-isomorphic basic support τ-tilting modules for block algebras of representation-finite and tame Schur algebras. Let $S(n, r)$ be a representation-finite or tame Schur algebra and B an indecomposable block algebra of $S(n, r)$. It is known from [Er] and [DEMN, Section 5] that the block algebra B is Morita equivalent to one of $\mathbb{F}$, $\mathcal{A}_m$, $\mathcal{D}_3$, $\mathcal{D}_4$, $\mathcal{R}_4$ and $\mathcal{H}_4$ (see Section 5.2 for the definitions). Then, we have

Main Theorem 5 (Theorem 5.2.2 and Lemma 5.2.3)**.** Let $\mathsf{s}\tau\text{-}\mathsf{tilt}\, B$ be the set of pairwise non-isomorphic basic support τ-tilting B-modules. Then,

B	$\mathcal{A}_m$	$\mathcal{D}_3$	$\mathcal{D}_4$	$\mathcal{R}_4$	$\mathcal{H}_4$
$\#\mathsf{s}\tau\text{-}\mathsf{tilt}\, B$	$\binom{2m}{m}$	28	114	88	96

,

where $\binom{x}{y}$ is the binomial coefficient.

Chapter 2

Preliminaries

In this chapter, we review the background of the representation theory of finite-dimensional algebras and the basic knowledge of quiver representations. We also explain in detail what τ-tilting theory is and the connection between τ-tilting theory and silting theory. In particular, we present several reduction theorems in this chapter.

A finite quiver $Q = (Q_0, Q_1, s, t)$ is a quadruple consisting of two finite sets: the vertex set Q_0 and the arrow set Q_1, and two maps: $s, t : Q_1 \to Q_0$ which associate to each arrow $\alpha \in Q_1$ its source $s(\alpha) \in Q_0$ and its target $t(\alpha) \in Q_0$, respectively. A non-trivial path in Q is a sequence $\alpha_1\alpha_2 \cdots \alpha_n$ ($n \geqslant 1$) of arrows which satisfies $s(\alpha_{i+1}) = t(\alpha_i)$ with $1 \leqslant i \leqslant n-1$. For a non-trivial path $\alpha_1\alpha_2 \cdots \alpha_n$, we set $s(\alpha_1\alpha_2 \cdots \alpha_n) := s(\alpha_1)$ and $t(\alpha_1\alpha_2 \cdots \alpha_n) := t(\alpha_n)$. Then, $\alpha_1\alpha_2 \cdots \alpha_n$ is called a cycle when $s(\alpha_1) = t(\alpha_n)$, and $\alpha_1\alpha_2 \cdots \alpha_n$ is called a loop when it is a cycle and $n = 1$. We denote by e_i the trivial path which satisfies $s(e_i) = t(e_i) = i$, for each vertex $i \in Q_0$.

Let K be an algebraically closed field. The path algebra KQ is the K-algebra whose basis is given by the set of all paths in Q and such that the multiplication of two paths w_1, w_2 is given by

$$w_1 \cdot w_2 := \begin{cases} \text{the composition } w_1w_2 & \text{if } t(w_1) = s(w_2), \\ 0 & \text{otherwise.} \end{cases}$$

We call the two-sided ideal of KQ generated by all arrows in Q_1 the arrow ideal of KQ and denote it by R_Q. A two-sided ideal I of KQ is said to be admissible if there exists $m \geqslant 2$ such that $(R_Q)^m \subseteq I \subseteq (R_Q)^2$. In this case, KQ/I is called a bound quiver algebra.

In this thesis, we always consider an associative finite-dimensional K-algebra A with an identity over an algebraically closed field K. We denote by A^{op} the opposite algebra of A. We denote by $\mathsf{mod}\ A$ the category of finitely generated right A-modules and by $\mathsf{proj}\ A$ (resp., $\mathsf{inj}\ A$) the category of finitely generated projective (resp., injective) right A-modules. For any $M \in \mathsf{mod}\ A$, we denote by $|M|$ the number of isomorphism classes of indecomposable direct summands of M.

We say that two K-algebras A and B are Morita equivalent if their module categories $\mathsf{mod}\ A$ and $\mathsf{mod}\ B$ are equivalent. Then, any K-algebra A is Morita equivalent to a bound quiver algebra KQ/I, where KQ is the path algebra of the Gabriel quiver $Q = Q_A$ of A and I is an admissible ideal of KQ. For each vertex $i \in Q$, we denote by e_i (same with the trivial path on i) the primitive idempotent of A associated with i, and by S_i (resp., P_i) the corresponding simple (resp., indecomposable projective) A-module. We often describe A-modules via their composition factors. For example, we denote the simple module S_i by i and then, $\begin{smallmatrix}1\\2\end{smallmatrix} = \begin{smallmatrix}S_1\\S_2\end{smallmatrix}$ is an indecomposable A-module M with a unique simple submodule S_2 such that $M/S_2 \simeq S_1$. We refer to [ASS] for more details.

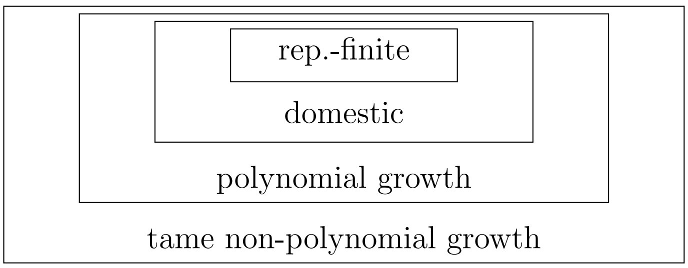
It is well-known from Drozd's Tame and Wild Theorem [Dro] that all finite-dimensional algebras A can be divided into two disjoint classes:

(1) A is tame if for any dimension d, there exists a finite number of $K[x]$-A-bimodules M_i $(1 \leqslant i \leqslant n_d)$, which are finitely generated and free as left $K[x]$-modules such that all but finitely many isomorphism classes of indecomposable right A-modules of dimension d, are of the form $K[x]/(x-\omega)\otimes_{K[x]} M_i$ with $\omega \in K$ and $i \in \{1, 2, \ldots, n_d\}$. More precisely, let $\mu_A(d)$ be the least number of $K[x]$-A-bimodules satisfying the above condition for dimension $d \geqslant 1$. We recall from [Sk3, Section 1.5] that

- A is representation-finite if and only if $\mu_A(d) = 0$ for any d.
- A is of domestic type if there is a constant C with $\mu_A(d) \leqslant C$ for any d.
- A is of polynomial growth type if there are positive integer m and constant C such that $\mu_A(d) \leqslant Cd^m$ for any d.

(2) A is wild if there is a finitely generated $K\langle X, Y\rangle$-A-bimodule M which is free over $K\langle X, Y\rangle$ and sends non-isomorphic indecomposable $K\langle X, Y\rangle$-modules via the functor $M \otimes_{K\langle X,Y\rangle} -$ to non-isomorphic indecomposable A-modules.

We have the following hierarchy and each of the inclusions is proper. In this thesis, a tame algebra always means a representation-infinite tame algebra.



We will need the following definitions.

Definition 2.0.1. Let Λ be a set. A partial order on Λ is a relation $\leq$ such that

(1) (reflexivity) $x \leq x$,

(2) (antisymmetry) $x \leq y$ and $y \leq x$ imply $x = y$,

(3) (transitivity) $x \leq y$ and $y \leq z$ imply $x \leq z$,

for all $x, y, z \in \Lambda$. We call $(\Lambda, \leq)$ a partially ordered set, or poset for short.

Definition 2.0.2. Let $(\Lambda, \leq)$ be a poset and $x, y \in \Lambda$. We say that y is covered by x if $y < x$ and there is no $z \in \Lambda$ with $y < z < x$. Then, the Hasse quiver $\mathcal{H}(\Lambda)$ of Λ consists of vertices representing the elements of Λ, and there is a unique arrow $x \to y$ from x to y if and only if y is covered by x.

Definition 2.0.3 ([B1, Definition 1.3], see also [HR1])**.** Let A be an algebra. An A-module M is called a tilting module if $|M| = |A|$, $\mathsf{Ext}_A^1(M, M) = 0$ and the projective dimension of M is at most one.

2.1 τ-tilting theory

For any right A-module M, we denote by $\mathsf{add}(M)$ (resp., $\mathsf{Fac}(M)$) the full subcategory of $\mathsf{mod}\ A$ whose objects are direct summands (resp., factor modules) of finite direct sums of copies of M. In order to give the definition of the Auslander-Reiten translation τ, we first recall the standard K-duality

$$D := \mathsf{Hom}_K(-, K) : \mathsf{mod}\ A \longleftrightarrow \mathsf{mod}\ A^{\mathsf{op}}$$

and the A-duality

$$(-)^* := \mathsf{Hom}_A(-, A) : \mathsf{proj}\ A \longleftrightarrow \mathsf{proj}\ A^{\mathsf{op}}.$$

Then, the Nakayama functor $\nu := D(-)^*$ is defined by the composition of D and $(-)^*$. It is well-known that the Nakayama functor ν induces equivalences of two categories

$$\mathsf{proj}\ A \underset{\nu^{-1}}{\overset{\nu}{\rightleftarrows}} \mathsf{inj}\ A\ ,$$

where $\nu^{-1} := \mathsf{Hom}_A(DA, -)$ is the quasi-inverse of ν. For any $M \in \mathsf{mod}\ A$ with a minimal projective presentation

$$P'' \xrightarrow{f_1} P' \xrightarrow{f_0} M \longrightarrow 0,$$

the Auslander-Reiten translation τM of M is defined by the following exact sequence

$$0 \longrightarrow \tau M \longrightarrow \nu P'' \xrightarrow{\nu f_1} \nu P'.$$

Definition 2.1.1 ([AIR, Definition 0.1])**.** Let $M \in \mathsf{mod}\ A$. Then,

(1) M is called τ-rigid if $\mathsf{Hom}_A(M, \tau M) = 0$.
(2) M is called τ-tilting if M is τ-rigid and $|M| = |A|$.
(3) M is called support τ-tilting if M is a τ-tilting (A/AeA)-module with respect to an idempotent e of A.

Corresponding to support τ-tilting modules, we may define support τ-tilting pairs. For any $M \in \mathsf{mod}\ A$ and $P \in \mathsf{proj}\ A$, the pair (M, P) is called a support τ-tilting pair if M is τ-rigid, $\mathsf{Hom}_A(P, M) = 0$ and $|M| + |P| = |A|$. Obviously, a pair (M, P) is a support τ-tilting pair if and only if M is a τ-tilting (A/AeA)-module and $P = eA$.

We denote by τ-$\mathsf{rigid}\ A$ (resp., $\mathsf{s}\tau$-$\mathsf{tilt}\ A$) the set of isomorphism classes of indecomposable τ-rigid (resp., basic support τ-tilting) A-modules.

Example 2.1.2. Let $A := K(\ 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2\ )/ < \alpha\beta, \beta\alpha >$. We denote by S_1, S_2 the simple A-modules and by P_1, P_2 the indecomposable projective A-modules. One may check that $\tau S_1 = S_2, \tau S_2 = S_1$ and $\tau P_1 = \tau P_2 = 0$. Then, we have $\tau\text{-rigid}\ A = \{P_1, P_2, S_1, S_2, 0\}$ and $\mathsf{s}\tau\text{-tilt}\ A = \{P_1 \oplus P_2, P_1 \oplus S_1, S_2 \oplus P_2, S_1, S_2, 0\}$. In particular, $P_1 \oplus P_2$, $P_1 \oplus S_1$, $S_2 \oplus P_2$ are τ-tilting modules and (S_1, P_2), (S_2, P_1) are support τ-tilting pairs.

We recall the following proposition to illustrate the relation between τ-rigid modules and τ-tilting modules.

Proposition 2.1.3 ([AIR, Theorem 0.2])**.** *Any τ-rigid A-module is a direct summand of some τ-tilting A-module.*

We recall the concept of left mutation, which is the core concept of τ-tilting theory. Before doing this, we need the definition of minimal left approximation. Let $\mathcal{C}$ be an additive category and X, Y objects of $\mathcal{C}$. A morphism $f : X \to Z$ with $Z \in \mathsf{add}(Y)$ is called a minimal left $\mathsf{add}(Y)$-approximation of X if f satisfies

- every $h \in \mathsf{Hom}_{\mathcal{C}}(Z, Z)$ that satisfies $h \circ f = f$ is an automorphism,
- $\mathsf{Hom}_{\mathcal{C}}(f, Z') : \mathsf{Hom}_{\mathcal{C}}(Z, Z') \longrightarrow \mathsf{Hom}_{\mathcal{C}}(X, Z')$ is surjective for any $Z' \in \mathsf{add}(Y)$,

where $\mathsf{add}(Y)$ is the category of all direct summands of finite direct sums of copies of Y.

Definition-Theorem 2.1.4 ([AIR, Definition 2.19, Theorem 2.30])**.** Let $T = M \oplus N$ be a basic support τ-tilting A-module with an indecomposable direct summand M satisfying $M \notin \mathsf{Fac}(N)$. We take an exact sequence with a minimal left $\mathsf{add}(N)$-approximation f:

$$M \xrightarrow{f} N' \longrightarrow \mathsf{coker}\ f \longrightarrow 0,$$

where $\mathsf{coker}\ f$ is the cokernel of f. We call $\mu_M^-(T) := (\mathsf{coker}\ f) \oplus N$ the left mutation of T with respect to M, which is again a basic support τ-tilting A-module. (The right mutation $\mu_M^+(T)$ can be defined dually.)

We may construct a directed graph $\mathcal{H}(\mathsf{s}\tau\text{-tilt}\ A)$ by drawing an arrow from T_1 to T_2 if T_2 is a left mutation of T_1. On the other hand, the set $\mathsf{s}\tau\text{-tilt}\ A$ has a poset structure with respect to the partial order $\leq$ defined as follows. For any $M, N \in \mathsf{s}\tau\text{-tilt}\ A$, let (M, P) and (N, Q) be their corresponding support τ-tilting pairs, respectively. We say that $N \leq M$ if $\mathsf{Fac}(N) \subseteq \mathsf{Fac}(M)$, or equivalently, $\mathsf{Hom}_A(N, \tau M) = 0$ and $\mathsf{add}(P) \subseteq \mathsf{add}(Q)$. Then,

Proposition 2.1.5 ([AIR, Theorem 2.33, Corollary 2.34])**.** *The directed graph $\mathcal{H}(\mathsf{s}\tau\text{-tilt}\ A)$ is exactly the Hasse quiver of the poset $\mathsf{s}\tau\text{-tilt}\ A$.*

The following statement implies that an algebra A is τ-tilting finite if we can find a finite connected component in $\mathcal{H}(\mathsf{s}\tau\text{-tilt}\ A)$.

Proposition 2.1.6 ([AIR, Corollary 2.38])**.** *If the Hasse quiver $\mathcal{H}(\mathsf{s}\tau\text{-tilt}\ A)$ contains a finite connected component Δ, then $\mathcal{H}(\mathsf{s}\tau\text{-tilt}\ A) = \Delta$.*

Example 2.1.7. Let $A = K(\, 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2\,)/ < \alpha\beta, \beta\alpha >$. We take an exact sequence with a minimal left $\mathsf{add}(P_1)$-approximation $f = \alpha \cdot -$ of P_2:

$$\begin{smallmatrix} e_2 \\ \beta \end{smallmatrix} \xrightarrow{f} \begin{smallmatrix} e_1 \\ \alpha \end{smallmatrix} \longrightarrow \mathsf{coker}\ f \longrightarrow 0.$$

Then, we have $\mathsf{coker}\ f = S_1$ and $\mu_{P_2}^-(A) = P_1 \oplus S_1$. Similarly, we can compute the left mutations step by step such that the Hasse quiver $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ A)$ is given as follows,

$$\begin{tikzcd}[row sep=small]
 & P_1 \oplus S_1 \arrow[r, ">"] & S_1 \arrow[dr, ">"] & \\
P_1 \oplus P_2 \arrow[ur, ">"] \arrow[dr, ">"] & & & 0\,. \\
 & S_2 \oplus P_2 \arrow[r, ">"] & S_2 \arrow[ur, ">"] &
\end{tikzcd}$$

Lastly, according to the duality $(-)^* = \mathsf{Hom}_A(-, A)$, we have

Proposition 2.1.8 ([AIR, Theorem 2.14])**.** *There exists a poset anti-isomorphism between* $\mathsf{s}\tau\text{-}\mathsf{tilt}\ A$ *and* $\mathsf{s}\tau\text{-}\mathsf{tilt}\ A^{\mathsf{op}}$.

2.2 Silting theory

We denote by $\mathsf{C}^{\mathsf{b}}(\mathsf{proj}\ A)$ the category of bounded complexes of projective A-modules and by $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\ A)$ the corresponding homotopy category which is triangulated. Besides, we denote by $\sim_h$ the homotopy equivalence in $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\ A)$. For any $T \in \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\ A)$, let $\mathsf{thick}\ T$ be the smallest full triangulated subcategory containing T, which is closed under cones, $[\pm 1]$, direct summands and isomorphisms.

Definition 2.2.1 ([AI, Definition 2.1])**.** A complex $T \in \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\ A)$ is called presilting if

$$\mathsf{Hom}_{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\ A)}(T, T[i]) = 0 \text{ for any } i > 0.$$

A presilting complex T is called silting if $\mathsf{thick}\ T = \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\ A)$. In particular, a silting complex T is called tilting if $\mathsf{Hom}_{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\ A)}(T, T[i]) = 0$ for any $i < 0$.

Similar to the left mutation of support τ-tilting modules, we recall the irreducible left silting mutation of silting complexes, see [AI, Definition 2.30]. Let $T = X \oplus Y$ be a basic silting complex in $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\ A)$ with an indecomposable summand X. We take a minimal left $\mathsf{add}(Y)$-approximation π and a triangle

$$X \xrightarrow{\pi} Z \longrightarrow \mathsf{cone}(\pi) \longrightarrow X[1],$$

where $\mathsf{cone}(\pi)$ is the mapping cone of π. Then, $\mathsf{cone}(\pi)$ is indecomposable and $\mu_X^-(T) := \mathsf{cone}(\pi) \oplus Y$ is again a basic silting complex in $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\ A)$, see [AI, Theorem 2.31]. We call $\mu_X^-(T)$ the irreducible left (silting) mutation of T with respect to X.

Definition 2.2.2. A complex in $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\ A)$ is called two-term if it is homotopy equivalent to a complex T concentrated in degree 0 and -1, i.e.,

$$(T^{-1} \xrightarrow{d_T^{-1}} T^0) = \cdots \longrightarrow 0 \longrightarrow T^{-1} \xrightarrow{d_T^{-1}} T^0 \longrightarrow 0 \longrightarrow \cdots.$$

We denote by $\mathsf{2\text{-}silt}\ A$ the set of isomorphism classes of basic two-term silting complexes in $\mathsf{K^b}(\mathsf{proj}\ A)$. Similarly, there is a partial order $\leq$ on the set $\mathsf{2\text{-}silt}\ A$ which is introduced by [AI, Theorem 2.11]. For T, $S \in \mathsf{2\text{-}silt}\ A$, we say that $S \leq T$ if $\mathsf{Hom}_{\mathsf{K^b}(\mathsf{proj}\ A)}(T, S[i]) = 0$ for any $i > 0$. Then, we denote by $\mathcal{H}(\mathsf{2\text{-}silt}\ A)$ the Hasse quiver of $\mathsf{2\text{-}silt}\ A$ which is compatible with the irreducible left mutation of silting complexes.

Proposition 2.2.3 ([AI, Lemma 2.25, Theorem 2.27]). *Let $T = (T^{-1} \to T^0) \in \mathsf{2\text{-}silt}\ A$. Then, we have $\mathsf{add}(A) = \mathsf{add}(T^0 \oplus T^{-1})$ and $\mathsf{add}(T^0) \cap \mathsf{add}(T^{-1}) = 0$.*

Suppose that $P_1, P_2, \ldots, P_n$ are pairwise non-isomorphic indecomposable projective A-modules. We denote by $[P_1], [P_2], \ldots, [P_n]$ the isomorphism classes of indecomposable complexes concentrated in degree 0. Clearly, the classes $[P_1], [P_2], \ldots, [P_n]$ in $\mathsf{K^b}(\mathsf{proj}\ A)$ form a standard basis of the Grothendieck group $\mathsf{K_0}(\mathsf{K^b}(\mathsf{proj}\ A))$. If a two-term complex T in $\mathsf{K^b}(\mathsf{proj}\ A)$ is written as

$$\left(\bigoplus_{i=1}^{n} P_i^{\oplus b_i} \longrightarrow \bigoplus_{i=1}^{n} P_i^{\oplus a_i} \right),$$

then the class $[T]$ can be identified by an integer vector

$$g(T) = (a_i - b_i)_{i \in \{1,2,\ldots,n\}} \in \mathbb{Z}^n,$$

which is called the g-vector of T. Then, we have the following statement.

Proposition 2.2.4 ([AIR, Theorem 5.5]). *A basic two-term silting complex T is uniquely determined by its g-vector $g(T)$.*

Example 2.2.5. Recall that $A = K(1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2)/ < \alpha\beta, \beta\alpha >$. Then,

$$T_1 = \cdots \longrightarrow 0 \longrightarrow P_1 \oplus P_2 \longrightarrow 0 \longrightarrow \cdots, \text{ and}$$
$$T_2 = \cdots \longrightarrow P_2 \xrightarrow{\binom{\alpha}{0}} P_1^{\oplus 2} \longrightarrow 0 \longrightarrow \cdots,$$

are basic two-term silting complexes. Moreover, we have $g(T_1) = (1, 1)$ and $g(T_2) = (2, -1)$.

Next, we explain the connection between τ-tilting theory and silting theory.

Theorem 2.2.6 ([AIR, Theorem 3.2]). *There exists a poset isomorphism between $\mathsf{s\tau\text{-}tilt}\ A$ and $\mathsf{2\text{-}silt}\ A$. More precisely, the bijection is given by mapping a two-term silting complex T to its 0-th cohomology $H^0(T)$, and the inverse is given by*

$$M \longmapsto (P'' \oplus P \xrightarrow{\binom{f}{0}} P'),$$

where (M, P) is the corresponding support τ-tilting pair and $P'' \xrightarrow{f} P' \longrightarrow M \longrightarrow 0$ is a minimal projective presentation of M.

One may easily find the bijection between $\mathsf{s\tau\text{-}tilt}\ A$ and $\mathsf{2\text{-}silt}\ A$ in the following example.

Example 2.2.7. Let $A = K(\, 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2\,)/ < \alpha\beta, \beta\alpha >$. Then, $\mathcal{H}(\mathsf{2\text{-}silt}\ A)$ is given by

$$\begin{tikzcd}
 & \begin{bmatrix} P_1 \xrightarrow{\beta} P_2 \\ \oplus \\ 0 \longrightarrow P_2 \end{bmatrix} \arrow[r] & \begin{bmatrix} P_1 \xrightarrow{\beta} P_2 \\ \oplus \\ P_1 \longrightarrow 0 \end{bmatrix} \arrow[dr] & \\
\begin{bmatrix} 0 \longrightarrow P_1 \\ \oplus \\ 0 \longrightarrow P_2 \end{bmatrix} \arrow[ur] \arrow[dr] & & & \begin{bmatrix} P_1 \longrightarrow 0 \\ \oplus \\ P_2 \longrightarrow 0 \end{bmatrix}. \\
 & \begin{bmatrix} 0 \longrightarrow P_1 \\ \oplus \\ P_2 \xrightarrow{\alpha} P_1 \end{bmatrix} \arrow[r] & \begin{bmatrix} P_2 \longrightarrow 0 \\ \oplus \\ P_2 \xrightarrow{\alpha} P_1 \end{bmatrix} \arrow[ur] &
\end{tikzcd}$$

We introduce a reduction theorem for complexes in the homotopy category $\mathsf{K^b}(\mathsf{proj}\ A)$, which will dramatically reduce the direct calculations in this thesis.

Lemma 2.2.8. *Let $\sim_h$ be the homotopy equivalence in $\mathsf{K^b}(\mathsf{proj}\ A)$. If $Y \neq 0$ and*

$$T_1 := (\, 0 \longrightarrow X \xrightarrow{\binom{1}{f}} X \oplus Y \xrightarrow{(-g\circ f, g)} Z \longrightarrow 0\,) \in \mathsf{K^b}(\mathsf{proj}\ A),$$

$$T_2 := (\, 0 \longrightarrow X \oplus Y \xrightarrow{\begin{pmatrix} f_1 & f_2 \\ 1 & g \\ h_1 & h_2 \end{pmatrix}} Z \oplus X \oplus M \longrightarrow 0\,) \in \mathsf{K^b}(\mathsf{proj}\ A),$$

then we have $T_1 \sim_h T_1^r$ and $T_2 \sim_h T_2^r$, where

$$T_1^r = (\, 0 \longrightarrow Y \xrightarrow{g} Z \longrightarrow 0\,) \in \mathsf{K^b}(\mathsf{proj}\ A),$$

$$T_2^r = (\, 0 \longrightarrow Y \xrightarrow{\begin{pmatrix} f_2 - f_1\circ g \\ h_2 - h_1 \circ g \end{pmatrix}} Z \oplus M \longrightarrow 0\,) \in \mathsf{K^b}(\mathsf{proj}\ A).$$

Proof. (1) We define $\varphi : T_1 \to T_1^r$ and $\psi : T_1^r \to T_1$ as follows,

$$\begin{tikzcd}
T_1: & 0 \arrow[r] & X \arrow[r, "\binom{1}{f}"] \arrow[d, shift left, "0"] & X \oplus Y \arrow[r, "(-g\circ f, g)"] \arrow[d, shift left, "(-f,1)"] & Z \arrow[r] \arrow[d, shift left, "1"] & 0 \\
T_1^r: & 0 \arrow[r] & 0 \arrow[r, "0"'] \arrow[u, dashed, shift left, "0"] & Y \arrow[r, "g"'] \arrow[u, dashed, shift left, "\binom{0}{1}"] & Z \arrow[r] \arrow[u, dashed, shift left, "1"] & 0
\end{tikzcd}.$$

Then, we have $\varphi \circ \psi = \mathrm{Id}_{T_1^r}$ and

$$\psi \circ \varphi = \left(0, \begin{pmatrix} 0 & 0 \\ -f & 1 \end{pmatrix}, 1\right) \sim_h \mathrm{Id}_{T_1},$$

because the difference $\mathrm{Id}_{T_1} - \psi \circ \varphi$ is null-homotopic as follows,

$$\begin{tikzcd}
T_1: & 0 \arrow[r] & X \arrow[r, "\binom{1}{f}"] \arrow[d, "1"] \arrow[dl, dotted, "0"'] & X \oplus Y \arrow[r, "(-g\circ f, g)"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ f & 0 \end{pmatrix}"] \arrow[dl, dotted, "(1,0)"'] & Z \arrow[r] \arrow[d, "0"] \arrow[dl, dotted, "0"'] & 0 \arrow[dl, dotted, "0"'] \\
T_1: & 0 \arrow[r] & X \arrow[r, "\binom{1}{f}"'] & X \oplus Y \arrow[r, "(-g\circ f, g)"'] & Z \arrow[r] & 0
\end{tikzcd}.$$

(2) We define $\varphi : T_2 \to T_2^r$ and $\psi : T_2^r \to T_2$ as follows,

$$\begin{tikzcd}
T_2: & 0 \arrow[r] & X\oplus Y \arrow[r, "\left(\begin{smallmatrix} f_1 & f_2 \\ 1 & g \\ h_1 & h_2 \end{smallmatrix}\right)"] \arrow[d, shift left, "{(0,1)}"] & Z\oplus X\oplus M \arrow[r] \arrow[d, shift left, "\left(\begin{smallmatrix} 1 & -f_1 & 0 \\ 0 & -h_1 & 1 \end{smallmatrix}\right)"] & 0 \\
T_2^r: & 0 \arrow[r] & Y \arrow[r, "\left(\begin{smallmatrix} f_2-f_1\circ g \\ h_2-h_1\circ g \end{smallmatrix}\right)"'] \arrow[u, dashed, shift left, "\left(\begin{smallmatrix} -g \\ 1 \end{smallmatrix}\right)"] & Z\oplus M \arrow[r] \arrow[u, dashed, shift left, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)"] & 0
\end{tikzcd}.$$

Then, we have $\varphi\circ\psi = \mathrm{Id}_{T_2^r}$ and

$$\psi\circ\varphi = \left(\left(\begin{smallmatrix} 0 & -g \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & -f_1 & 0 \\ 0 & 0 & 0 \\ 0 & -h_1 & 1 \end{smallmatrix}\right)\right) \sim_h \mathrm{Id}_{T_2}.$$

In fact, the difference $\mathrm{Id}_{T_2} - \psi\circ\varphi$ is null-homotopic as follows,

$$\begin{tikzcd}
T_2: & 0 \arrow[r] & X\oplus Y \arrow[r, "\left(\begin{smallmatrix} f_1 & f_2 \\ 1 & g \\ h_1 & h_2 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 1 & g \\ 0 & 0 \end{smallmatrix}\right)"'] \arrow[dl, dotted, "0"'] & Z\oplus X\oplus M \arrow[r] \arrow[d, "\left(\begin{smallmatrix} 0 & f_1 & 0 \\ 0 & 1 & 0 \\ 0 & h_1 & 0 \end{smallmatrix}\right)"'] \arrow[dl, dotted, "\left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)"'] & 0 \arrow[dl, dotted, "0"'] \\
T_2: & 0 \arrow[r] & X\oplus Y \arrow[r, "\left(\begin{smallmatrix} f_1 & f_2 \\ 1 & g \\ h_1 & h_2 \end{smallmatrix}\right)"'] & Z\oplus X\oplus M \arrow[r] & 0
\end{tikzcd}.$$

Therefore, we have $T_1 \sim_h T_1^r$ and $T_2 \sim_h T_2^r$. □

2.3 τ-tilting finite algebras

In this section, we will introduce the main object we are interested in. Recall that A is a finite-dimensional basic algebra over an algebraically closed field K.

Definition 2.3.1. We call A a τ-tilting finite algebra if there are only finitely many isomorphism classes of basic τ-tilting A-modules. Otherwise, A is called τ-tilting infinite.

Moreover, we have some equivalent conditions for A to be τ-tilting finite.

Proposition 2.3.2 ([DIJ, Corollary 2.9])**.** *An algebra A is τ-tilting finite if and only if one of (equivalently, any of) the sets τ-$\mathsf{rigid}\ A$, $\mathsf{s}\tau$-$\mathsf{tilt}\ A$ and 2-$\mathsf{silt}\ A$ is a finite set.*

A typical example of τ-tilting finite algebras is the class of representation-finite algebras. In particular, we have known the number $\#\mathsf{s}\tau$-$\mathsf{tilt}\ K\Delta$ for a representation-finite path algebra $K\Delta$. We recall this result as follows.

Proposition 2.3.3 ([Ad2, Proposition 1.6], [ONFR, Theorem 1])**.** *Let $K\Delta$ be a path algebra with a finite quiver Δ whose underlying graph is one of Dynkin diagrams of type $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$. Then, $\#\mathsf{s}\tau$-$\mathsf{tilt}\ K\Delta$ is independent of the orientation of Δ and*

Δ	$\mathbb{A}_n$	$\mathbb{D}_n (n \geqslant 4)$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$
$\#\mathsf{s}\tau$-$\mathsf{tilt}\ K\Delta$	$\frac{1}{n+2}\binom{2n+2}{n+1}$	$\left[\begin{smallmatrix} 2n-1 \\ n-1 \end{smallmatrix}\right]$	833	4160	25080

,

where $\binom{x}{y}$ is the binomial coefficient and $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \frac{x+y}{x}\binom{x}{y}$.

Since the class of τ-tilting infinite algebras is infinite, we are interested in the minimal cases among all τ-tilting infinite algebras. We recall that an algebra B is said to be a quotient (or quotient algebra) of an algebra A if there exists a surjective K-algebra homomorphism $\phi : A \to B$.

Definition 2.3.4. An algebra A is called minimal τ-tilting infinite if A is τ-tilting infinite, but any proper quotient algebra of A is τ-tilting finite.

It is known that any path algebra $K\Delta$ with a finite quiver Δ whose underlying graph is one of Euclidean diagrams of type $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{D}}_n (n \geqslant 4)$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$, is minimal τ-tilting infinite. Besides, Mousavand also introduced this notion independently in his recent work [Mo], where the author is trying to give a complete classification of minimal representation-infinite algebras in terms of τ-tilting finiteness.

We recall two reduction theorems for the τ-tilting finiteness of an algebra A, which will imply the importance of minimal τ-tilting infinite algebras.

Proposition 2.3.5 ([DIJ, Theorem 4.2], [DIRRT, Theorem 5.12])**.** *If A is τ-tilting finite,*

(1) the quotient algebra A/I is τ-tilting finite for any two-sided ideal I of A,

(2) the idempotent truncation eAe is τ-tilting finite for any idempotent e of A.

In Proposition 2.3.5 (1), we may reduce the question on A to the question on A/I if we take a special two-sided ideal I of A. This technical method is powerful and it is provided by Eisele, Janssens and Raedschelders [EJR]. We recall this method as follows.

Proposition 2.3.6 ([EJR, Theorem 1])**.** *Let I be a two-sided ideal generated by central elements which are contained in the Jacobson radical of A. Then, there exists a poset isomorphism between $\mathsf{s}\tau\text{-}\mathsf{tilt}\ A$ and $\mathsf{s}\tau\text{-}\mathsf{tilt}\ (A/I)$.*

We notice that the τ-rigid-brick correspondence is also useful for determining the τ-tilting finiteness of an algebra A. Recall that K is assumed to be an algebraically closed field. Then, M is called a brick if $\mathsf{End}_A(M) = K$. We denote by $\mathsf{brick}\ A$ the set of isomorphism classes of bricks in $\mathsf{mod}\ A$.

Proposition 2.3.7 ([DIJ, Theorem 4.2])**.** *Let A be a finite-dimensional algebra. Then, A is τ-tilting finite if and only if the set $\mathsf{brick}\ A$ is finite.*

When we compute the number $\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ A$ in some cases, Proposition 2.3.6 may greatly reduce the direct calculations of left mutations. For example, let $A := KQ/I$ with

$$Q : \begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[r, shift left, "\mu"] & 2 \arrow[l, shift left, "\nu"] \end{tikzcd} \text{ and } I :< \alpha^2 - \mu\nu, \nu\alpha\mu >.$$

We denote the indecomposable projective A-modules by P_1 and P_2. Then, we have

$$P_1 = \begin{tikzcd}[row sep=tiny, column sep=tiny] & & e_1 \arrow[dl, dash] \arrow[dr, dash] & \\ & \alpha \arrow[dl, dash] \arrow[dr, dash] & & \mu \arrow[dl, dash] \\ \alpha\mu \arrow[dr, dash] & & \alpha^2 \arrow[dl, dash] \arrow[dr, dash] & \\ & \alpha^3 \arrow[dr, dash] & & \alpha^2\mu \arrow[dl, dash] \\ & & \alpha^4 & \end{tikzcd} \simeq \begin{tikzcd}[row sep=tiny, column sep=tiny] & & 1 \arrow[dl, dash] \arrow[dr, dash] & \\ & 1 \arrow[dl, dash] \arrow[dr, dash] & & 2 \arrow[dl, dash] \\ 2 \arrow[dr, dash] & & 1 \arrow[dl, dash] \arrow[dr, dash] & \\ & 1 \arrow[dr, dash] & & 2 \arrow[dl, dash] \\ & & 1 & \end{tikzcd} \text{ and } P_2 = \begin{tikzcd}[row sep=tiny, column sep=tiny] & e_2 \arrow[d, dash, dotted] & \\ & \nu \arrow[dl, dash] \arrow[dr, dash] & \\ \nu\alpha \arrow[dr, dash] & & \nu\mu \arrow[dl, dash] \\ & \nu\alpha^2 \arrow[d, dash] & \\ & \nu\alpha^2\mu & \end{tikzcd} \simeq \begin{tikzcd}[row sep=tiny, column sep=tiny] & 2 \arrow[d, dash, dotted] & \\ & 1 \arrow[dl, dash] \arrow[dr, dash] & \\ 1 \arrow[dr, dash] & & 2 \arrow[dl, dash] \\ & 1 \arrow[d, dash, dotted] & \\ & 2 & \end{tikzcd}.$$

Now, we may focus on the central element $\mu\nu + \nu\mu$ of A, and consider the quotient algebra $\widetilde{A} := A/ < \mu\nu + \nu\mu >$. Let $\widetilde{P_1}$ and $\widetilde{P_2}$ be the indecomposable projective $\widetilde{A}$-modules. Then,

$$\widetilde{P_1} = \begin{tikzcd}[row sep=tiny, column sep=tiny] & e_1 \arrow[dl, dash] \arrow[dr, dash] & \\ \alpha \arrow[d, dash] & & \mu \\ \alpha\mu & & \end{tikzcd} \simeq \begin{tikzcd}[row sep=tiny, column sep=tiny] & 1 \arrow[dl, dash] \arrow[dr, dash] & \\ 1 \arrow[d, dash, dotted] & & 2 \\ 2 & & \end{tikzcd} \text{ and } \widetilde{P_2} = \begin{tikzcd}[row sep=tiny] e_2 \arrow[d, dash, dotted] \\ \nu \arrow[d, dash, dotted] \\ \nu\alpha \end{tikzcd} \simeq \begin{tikzcd}[row sep=tiny] 2 \arrow[d, dash, dotted] \\ 1 \arrow[d, dash, dotted] \\ 1 \end{tikzcd}.$$

By direct calculation, one may find that the Hasse quiver $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ \widetilde{A})$ is as follows,

$$\begin{tikzcd} \begin{smallmatrix}1&\\1&2\\2&\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\1\end{smallmatrix} \arrow[r] \arrow[d] & \begin{smallmatrix}2&\\1&2\\1&\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\1\end{smallmatrix} \arrow[r] & \begin{smallmatrix}2&\\1&2\\1&\end{smallmatrix} \oplus 2 \arrow[r] & 2 \arrow[d] \\ \begin{smallmatrix}1&\\1&2\\2&\end{smallmatrix} \oplus \begin{smallmatrix}1\\1\\2\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1\\1\end{smallmatrix} \oplus \begin{smallmatrix}1\\1\\2\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1\\1\end{smallmatrix} \arrow[r] & 0 \end{tikzcd}.$$

Thus, we deduce that $\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ \widetilde{A} = 8$. By Proposition 2.3.6, we have $\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ A = 8$.

Next, we consider $\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ A$ for an arbitrary τ-tilting finite algebra A. We say that a right A-module M is support-rank s if there exist exactly s nonzero primitive orthogonal idempotents $e_1, e_2, \ldots, e_s$ of A such that $Me_i \neq 0$. Then, we denote by $a_s(A)$ the number of pairwise non-isomorphic basic support τ-tilting A-modules with support-rank s for $0 \leqslant s \leqslant |A|$. It is obvious that $a_0(A) = 1$ and $a_1(A) = |A|$ since it is the local algebra case. Then, we have

$$\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ A = \sum_{s=0}^{|A|} a_s(A).$$

Note that $a_{|A|}(A)$ is just the number of pairwise non-isomorphic basic τ-tilting A-modules.

Let M be a support τ-tilting A-module. According to Definition 2.1.1, we may assume that M is a τ-tilting B-module for

$$B := A/ < 1 - e_1 - e_2 - \cdots - e_s >,$$

with some nonzero orthogonal idempotents $e_1, e_2, \ldots, e_s$ of A. By [AIR, Proposition 2.2], τ-tilting B-modules are precisely sincere support τ-tilting B-modules, so that the support-rank of M is equal to $|M|$. Thus, we have

Proposition 2.3.8 ([Ad2, Proposition 1.8]). *Let M be a support τ-tilting A-module and $0 \leqslant s \leqslant |A|$. Then, M is support-rank s if and only if $|M| = s$.*

Although the number $\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ A$ can be determined for some special classes of algebras, such as path algebras of Dynkin type in Proposition 2.3.3, preprojective algebras of Dynkin type [Mi], etc., but it is not easy to find the number $\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ A$ for general cases.

We explain the reasons as follows. If A is a path algebra of Dynkin type, one can find some recursive relations between $a_{s-1}(A)$ and $a_s(A)$ such that we get a formula to calculate $\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ A$. If A is a preprojective algebra of Dynkin type Δ, we may construct a one-to-one correspondence between (pairwise non-isomorphic basic) support τ-tilting A-modules and elements in the Weyl group W_Δ associated with Δ. Since the number of elements in W_Δ is known, we get the number $\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ A$.

However, it is difficult to find such a recursive relation or such a one-to-one correspondence in general, so that we can only determine the number $\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ A$ by direct computation. We have tried to construct recursive relations between $a_{s-1}(A)$ and $a_s(A)$ for some examples. Here, we present a representation-finite tilted algebra as an example, and the proof is given in the Appendix A.2.

Example 2.3.9. Let Λ_3 be the path algebra of $\circ \longleftarrow \circ \longrightarrow \circ$ and $\Lambda_n := KQ_n/I$ $(n \geqslant 4)$ the algebra presented by the following quiver Q_n and $I = <\alpha\mu - \beta\nu>$,

$$Q_n:\ \begin{tikzcd}[row sep=small] & 2 \arrow[dr, "\mu"] & & & & & \\ 1 \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] \arrow[rr, dotted, no head] & & 4 \arrow[r] & 5 \arrow[r] & \cdots \arrow[r] & n-1 \arrow[r] & n \\ & 3 \arrow[ur, "\nu"'] & & & & & \end{tikzcd}.$$

Then, based on the symbols above, we have

(1) $a_n(\Lambda_n) = a_{n-1}(\Lambda_n) - \frac{1}{n-2}\binom{2n-6}{n-3}$.

(2) $a_{n-1}(\Lambda_n) = a_{n-1}(\Lambda_{n-1}) + \frac{3n-7}{2n-4}\binom{2n-4}{n-3} + 2\cdot\frac{(5n-11)\cdot(2n-6)!}{(n-3)!\cdot(n-1)!} + \sum\limits_{i=4}^{n-1} a_{i-1}(\Lambda_{i-1})\cdot\frac{1}{n-i+1}\binom{2(n-i)}{n-i}$.

(3) $a_{n-2}(\Lambda_n) = a_{n-2}(\Lambda_{n-1}) + a_{n-3}(\Lambda_n) + \frac{1}{n-2}\binom{2n-6}{n-3}$.

(4) $a_s(\Lambda_n) = a_s(\Lambda_{n-1}) + a_{s-1}(\Lambda_n)$ for any $1 \leqslant s \leqslant n-3$.

These recursive formulas enable us to compute $\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ \Lambda_n$ step by step. For example,

$a_s(\Lambda_n)$ \ s / n	0	1	2	3	4	5	6	7	8	9	$\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ \Lambda_n$
4	1	4	10	16	15						46
5	1	5	15	33	54	52					160
6	1	6	21	54	113	192	187				574
7	1	7	28	82	195	401	700	686			2100
8	1	8	36	118	313	714	1456	2592	2550		7788
9	1	9	45	163	476	1190	2646	5307	9702	9570	29172

We show that Λ_n is a tilted algebra of Dynkin type $\mathbb{D}_n$ and therefore, Λ_n is representation-finite following [ASS, VIII. Lemma 3.2]. Let $\vec{D}_n$ $(n \geqslant 4)$ be a path algebra with quiver:

$$\begin{tikzcd}[row sep=small] 1 \arrow[dr] & & & & & \\ & 3 \arrow[r] & 4 \arrow[r] & \cdots \arrow[r] & n-1 \arrow[r] & n \\ 2 \arrow[ur] & & & & & \end{tikzcd}.$$

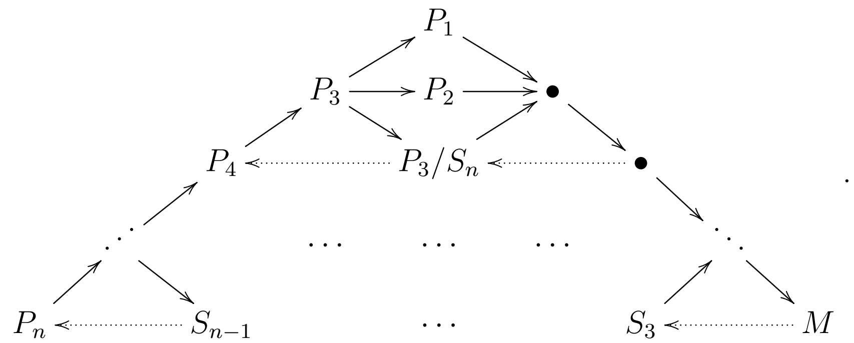
Then, the indecomposable projective $\vec{D}_n$-modules are displayed as follows,

$$P_1 = \begin{smallmatrix} 1 \\ 3 \\ 4 \\ \vdots \\ n \end{smallmatrix},\ P_2 = \begin{smallmatrix} 2 \\ 3 \\ 4 \\ \vdots \\ n \end{smallmatrix},\ P_3 = \begin{smallmatrix} 3 \\ 4 \\ \vdots \\ n \end{smallmatrix},\ P_4 = \begin{smallmatrix} 4 \\ \vdots \\ n \end{smallmatrix},\ \dots,\ P_{n-1} = \begin{smallmatrix} n-1 \\ n \end{smallmatrix},\ P_n = \begin{smallmatrix} n \end{smallmatrix}.$$

By Definition-Theorem 2.1.4, it is easy to find that

$$\mu^-_{P_3}(\vec{D}_n) = P_1 \oplus P_2 \oplus M \oplus P_4 \oplus \cdots \oplus P_{n-1} \oplus P_n, \text{ where } M = \begin{smallmatrix} 1 \ 2 \\ 3 \\ 4 \\ \vdots \\ n \end{smallmatrix},$$

and $\mu^-_{P_3}(\vec{D}_n)$ is a τ-tilting $\vec{D}_n$-module. Since tilting modules (see Definition 2.0.3) and τ-tilting modules coincide over a path algebra, $\mu^-_{P_3}(\vec{D}_n)$ is a tilting $\vec{D}_n$-module. Then, we have $\Lambda_n = \mathsf{End}_{\vec{D}_n}(\mu^-_{P_3}(\vec{D}_n))$. To see this, we observe the following sequence in the Auslander-Reiten quiver $\Gamma(\mathsf{mod}\ \vec{D}_n)$ of $\vec{D}_n$,



We define two non-zero morphisms $\alpha : P_1 \to M$ and $\mu : P_4 \to P_1$, then $\alpha\mu : P_4 \to M$. Similarly, we define β and $-\nu$ by $P_2 \to M$ and $P_4 \to P_2$, respectively. Since the composition $P_4 \to P_3/S_n \to M$ is zero, we have $\alpha\mu = \beta\nu$. (In fact, α, μ, β, ν are unique non-zero morphisms, and there are no non-zero morphisms from P_3/S_n to M.)

When we compute the left mutation of support τ-tilting modules, we usually start at A and end at 0 since A is the maximal element in $\mathsf{s}\tau\text{-}\mathsf{tilt}\ A$. As we explained before, we divide the whole calculation into the calculations for different support-ranks. We use the following example to illustrate our method.

Example 2.3.10. Let $A := KQ/I$ be the bound quiver algebra given by

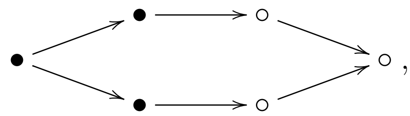
$$Q:\ \begin{tikzcd} 1 \arrow[r, shift left, "\alpha_1"] & 3 \arrow[l, shift left, "\beta_1"] \arrow[r, shift left, "\beta_3"] \arrow[d, shift left, "\beta_2"] & 4 \arrow[l, shift left, "\alpha_3"] \\ & 2 \arrow[u, shift left, "\alpha_2"] & \end{tikzcd} \text{ and } I : \left\langle \begin{matrix} \alpha_1\beta_1, \alpha_2\beta_2, \beta_2\alpha_2, \alpha_3\beta_1, \alpha_3\beta_2, \\ \alpha_1\beta_3, \alpha_2\beta_3, \alpha_3\beta_3, \beta_3\alpha_3, \alpha_2\beta_1\alpha_1\beta_2, \end{matrix} \right\rangle.$$

Since the support τ-tilting A-module with support-rank 0 is unique, we have $a_0(A) = 1$. Since each simple A-module S_i is an $A/A(1-e_i)A$-module and $A/A(1-e_i)A \simeq \mathbb{F}$, we observe that the support τ-tilting A-modules with support-rank 1 are exactly the simple A-modules. Then, $a_1(A) = |A| = 4$.

Let M be a support τ-tilting A-module with support-rank 2, and with supports e_i and e_j $(i \neq j)$. Then, M becomes a τ-tilting A/J-module with $J =< 1 - e_i - e_j >$. We denote by $\mathsf{b}_{i,j}$ the number of τ-tilting A/J-modules. For example, if $(i, j) = (1, 3)$, then

$$A/J = K\left(1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 3 \right) / \langle \alpha_1\beta_1 \rangle.$$

Note that $\beta_1\alpha_1$ is a central element of A/J, we may apply Proposition 2.3.6 and Example 2.1.7 to show that the Hasse quiver $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ A/J)$ is displayed as follows,



where we denote by $\bullet$ τ-tilting A/J-modules and by $\circ$ other support τ-tilting (but not τ-tilting) A/J-modules. Hence, we deduce that $\mathsf{b}_{1,3} = 3$. Similarly, we have

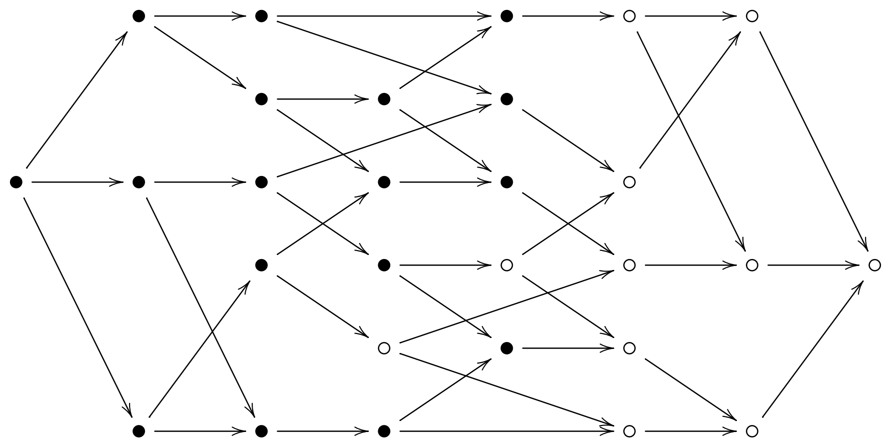
(i,j)	$(1,2)$	$(1,4)$	$(2,3)$	$(2,4)$	$(3,4)$
$\mathsf{b}_{i,j}$	1	1	3	1	3

This implies that $a_2(A) = 12$.

Let N be a support τ-tilting A-module with support-rank 3. Then, N becomes a τ-tilting A/L_j-module with $L_j =< e_j >$, where e_j is the only one non-zero primitive idempotent satisfying $Ne_j = 0$. We denote by d_j the number of τ-tilting A/L_j-modules. For example, if $j = 4$, then

$$A/L_4 = K\left(1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 3 \underset{\alpha_2}{\overset{\beta_2}{\rightleftarrows}} 2 \right) / \langle \alpha_1\beta_1, \alpha_2\beta_2, \beta_2\alpha_2, \alpha_2\beta_1\alpha_1\beta_2 \rangle.$$

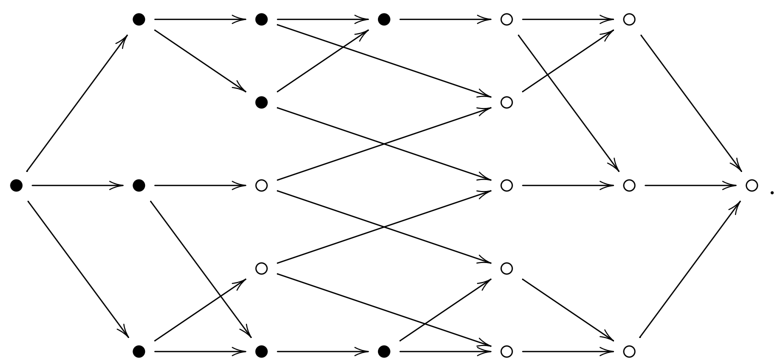
Similar to the above, we compute the left mutations by hand to show that the Hasse quiver $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ A/L_4)$ is as follows,



We deduce that $\mathsf{d}_4 = 17$. If $j = 2$, then

$$A/L_2 = K\left(1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 3 \underset{\alpha_3}{\overset{\beta_3}{\rightleftarrows}} 4 \right) / \langle \alpha_1\beta_1, \alpha_3\beta_1, \alpha_1\beta_3, \beta_3\alpha_3, \alpha_3\beta_3 \rangle,$$

and the Hasse quiver $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ A/L_2)$ is shown as follows by direct calculation,



We deduce that $\mathsf{d}_2 = 9$. Similarly, we have $\mathsf{d}_1 = 9$ and $\mathsf{d}_3 = 1$. Therefore, $a_3(A) = 36$.

We may compute $a_4(A)$ by hand, because the support τ-tilting A-modules with support-rank 4 are just τ-tilting A-modules, which can be obtained by the left mutations starting with A. See Appendix A.3 for a complete list of τ-tilting A-modules and one may easily construct the part of $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ A)$ consisting of all τ-tilting A-modules. Thus, $a_4(A) = 61$.

Finally, we conclude that $\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ A = 1 + 4 + 12 + 36 + 61 = 114$.

2.4 Tilting theory and derived equivalence

We briefly review the connection between tilting theory and τ-tilting theory. We mention that the definition of tilting modules is given in Definition 2.0.3. Then, it is shown in [AIR] that any tilting module is a τ-tilting module, and any faithful τ-tilting module is a tilting module. Here, a right A-module M is called *faithful* if the annihilator $\mathsf{ann}(M) = \{x \in A \mid Mx = 0\}$ is zero.

Also, we briefly introduce the connection between tilting theory and derived equivalence. Let $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\ A)$ be the derived category of bounded complexes of modules from $\mathsf{mod}\ A$, which is the localization of the homotopy category $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\ A)$ with respect to quasi-isomorphisms. Then, $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\ A)$ is also a triangulated category. We recall that two algebras A and B are said to be derived equivalent if their derived categories $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\ A)$ and $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\ B)$ are equivalent as triangulated categories.

It is worth mentioning that tilting modules induce an essential class of derived equivalence of algebras. Let M be a tilting A-module, the endomorphism algebra $B = \mathsf{End}_A\ M$ is called a tilted algebra of A. In this case, Happel [Ha, Corollary 1.7] showed that the algebras A and B are derived equivalent. More generally, we have

Proposition 2.4.1 ([Ric, Theorem 6.4])**.** *Let A and B be two algebras. Then, A and B are derived equivalent if and only if*

$$B \simeq \mathsf{End}_{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\ A)}\ (T)$$

for a tilting complex T in $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\ A)$.

Chapter 3

Algebras With Two Simple Modules

Since local algebras, i.e., algebras with only one simple module, are always τ-tilting finite, the class of algebras with exactly two simple modules (up to isomorphism) is fundamental in τ-tilting theory. In this chapter, we focus on this class of algebras and call them two-point algebras.

Two-point algebras are also fundamental if we consider the representation type of general algebras, so that the representation type of two-point algebras has been determined for many years. We may review these results here: the maximal representation-finite two-point algebras are classified by Bongartz and Gabriel [BoG], the tame two-point algebras are classified by several authors in [BH], [DG], [Gei], [Han] and the minimal wild two-point algebras are classified by Han [Han].

In [Han], a minimal wild algebra A is called strongly minimal wild if there is no wild algebra B with $|B| < |A|$ and a fully faithful exact functor $\mathcal{F} : \mathsf{mod}\ B \to \mathsf{mod}\ A$. It is obvious that any wild algebra A admits a strongly minimal wild algebra B with a fully faithful exact functor $\mathcal{F} : \mathsf{mod}\ B \to \mathsf{mod}\ A$. Notice that strongly minimal wild two-point algebras are also classified by Han [Han] and this class coincides with the class of minimal wild two-point algebras, except for an exact special case.

On the other hand, if B is τ-tilting infinite and there is a fully faithful functor $\mathcal{F} : \mathsf{mod}\ B \to \mathsf{mod}\ A$, then A is also τ-tilting infinite since $\mathcal{F}$ sends a brick in $\mathsf{mod}\ B$ to a brick in $\mathsf{mod}\ A$, see Proposition 2.3.7. Therefore, one wants to give a complete classification of minimal τ-tilting infinite two-point algebras. With this motivation in mind, we determine the τ-tilting finiteness for (strongly) minimal wild two-point algebras. According to our result Theorem 3.2.3, most of (strongly) minimal wild two-point algebras are τ-tilting finite. We mention that we do not know the τ-tilting finiteness for an arbitrary wild two-point algebra which has a τ-tilting finite minimal wild two-point algebra as a quotient algebra.

Toward the complete classification of τ-tilting finite two-point algebras, we also have to consider tame two-point algebras. However, it is difficult at this moment to give a complete result on tame two-point algebras, because the tameness of two-point algebras depends on the technique called *degeneration*, and it is still open to finding the relation between τ-tilting finiteness and degeneration. We may give a partial result on tame two-point algebras. We recall from [Han] (see Proposition 3.2.1 of this chapter) that all tame two-point algebras can degenerate to a finite set (Table T in [Han], see also Appendix A.2) of two-point algebras. Then, we check the τ-tilting finiteness for algebras in Table T.

We point out that Aihara-Kase [AK] and Kase [K] have got some interesting results. For example, Kase [K, Theorem 6.1] showed that for any natural numbers s and t, we can find a τ-tilting finite two-point algebra A such that the Hasse quiver $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ A)$ is of the following form

$$\begin{tikzcd}[row sep=small]
 & \triangle_1 \arrow[r] & \triangle_2 \arrow[r] & \cdots \arrow[r] & \triangle_s \arrow[dr] & \\
A \arrow[ur] \arrow[dr] & & & & & 0. \\
 & \square_1 \arrow[r] & \square_2 \arrow[r] & \cdots \arrow[r] & \square_t \arrow[ur] &
\end{tikzcd}$$

In this case, we say that $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ A)$ is of type $\mathcal{H}_{s,t}$ and it is easy to find $\mathcal{H}_{s,t} \simeq \mathcal{H}_{t,s}$. Besides, it is well-known that the Kronecker algebra $K(\circ \rightrightarrows \circ)$ is τ-tilting infinite (we present a proof in Lemma 3.1.1 for the convenience of readers).

In the first section of this chapter, we explain our strategy for reducing (strongly) minimal wild two-point algebras and algebras in Table T to a small set of two-point algebras. Then, we determine the τ-tilting finiteness of this small set. In the second section, we determine the τ-tilting finiteness for all (strongly) minimal wild two-point algebras and all algebras in Table T. We also observe that our results are useful to determine the τ-tilting finiteness for several other classes of algebras, such as tame two-point distributive algebras [Gei], two-point symmetric special biserial algebras [AIP] and so on. We have given some applications at the end of this chapter.

3.1 A small set of two-point algebras

Let A be a finite-dimensional algebra over an algebraically closed field K. We denote by $\mathsf{rad}(A)$ the Jacobson radical of A and by $C(A)$ the center of A. As explained in Section 2.3, although A has a complicated structure, its quotient algebra

$$\widetilde{A} := A/ < C(A) \cap \mathsf{rad}(A) >$$

may have a simpler structure. Moreover, by Proposition 2.3.6, we know that $\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ A = \#\mathsf{s}\tau\text{-}\mathsf{tilt}\ \widetilde{A}$ and the Hasse quivers $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ A)$ and $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ \widetilde{A})$ are of the same type. Then, by using this strategy and Proposition 2.3.5, we can restrict (strongly) minimal wild two-point algebras and algebras in Table T (except for three cases: W_4, T_{20} and T_{21}) to a small list (i.e., Table Λ) of two-point algebras. (In Table Λ, an algebra Λ_i is just the bound quiver algebra KQ/I_i, where I_i is the admissible ideal generated by the relation (i).)

Thus, as a preparation for proving our main results in this chapter, we shall determine the τ-tilting finiteness of Λ_i in Table Λ. We remark that $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ \Lambda_1)$ is of type $\mathcal{H}_{1,2}$ and $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ \Lambda_6)$ is of type $\mathcal{H}_{2,2}$, see Example 2.1.7.

Table 3.1: Table Λ

$\Lambda_1 = K\left(1 \longrightarrow 2\right)$;

$\Lambda_2 = K\left(1 \rightrightarrows 2\right)$;

$Q: 1 \xrightarrow{\mu} 2 \circlearrowleft \beta$

(3) $\beta^2 = 0$;
(4) $\beta^3 = 0$;

$Q: \alpha \circlearrowright 1 \xrightarrow{\mu} 2 \circlearrowleft \beta$

(5) $\alpha^2 = \beta^2 = 0$;

$Q: 1 \underset{\nu}{\overset{\mu}{\rightleftarrows}} 2$

(6) $\mu\nu = \nu\mu = 0$;

$Q: \alpha \circlearrowright 1 \underset{\nu}{\overset{\mu}{\rightleftarrows}} 2$

(7) $\alpha^2 = \mu\nu = \nu\mu = \nu\alpha = 0$;
(8) $\alpha^2 = \mu\nu = \nu\mu = \nu\alpha\mu = 0$;
(9) $\alpha^3 = \mu\nu = \nu\mu = \nu\alpha = 0$;
(10) $\alpha^3 = \mu\nu = \nu\mu = \nu\alpha\mu = \nu\alpha^2\mu = 0$;

$Q: \alpha \circlearrowright 1 \underset{\nu}{\overset{\mu}{\rightleftarrows}} 2 \circlearrowleft \beta$

(11) $\alpha^2 = \beta^2 = \mu\nu = \nu\mu = \alpha\mu = \beta\nu = 0$;
(12) $\alpha^2 = \beta^2 = \mu\nu = \nu\mu = \beta\nu = \nu\alpha = \alpha\mu\beta = 0$.

Lemma 3.1.1. *The Kronecker algebra Λ_2 is minimal τ-tilting infinite.*

Proof. It is easy to check that $M_k = K \underset{1}{\overset{k}{\rightrightarrows}} K$ with $k \in K$ is a brick in $\mathsf{mod}\,\Lambda_2$. Since the family $(M_k)_{k\in K}$ consists of infinitely many pairwise non-isomorphic bricks, Λ_2 is τ-tilting infinite by Proposition 2.3.7. Then, the minimality is obvious. $\square$

Lemma 3.1.2. *The two-point algebras Λ_3 and Λ_4 are τ-tilting finite.*

Proof. Since Λ_3 is a quotient algebra of Λ_4, by Proposition 2.3.5, it suffices to show that Λ_4 is τ-tilting finite. We show that the poset $\mathsf{2\text{-}silt}\,\Lambda_4$ has a finite connected component and hence, it exhausts all two-term silting complexes in $\mathsf{K^b}(\mathsf{proj}\,\Lambda_4)$ by Proposition 2.1.6 and Theorem 2.2.6. Then, Λ_4 is τ-tilting finite following from Proposition 2.3.2. Let P_1 and P_2 be the indecomposable projective Λ_4-modules. We have

$$P_1 = \begin{smallmatrix} e_1 \\ \mu \\ \mu\beta \\ \mu\beta^2 \end{smallmatrix} \simeq \begin{smallmatrix} 1 \\ 2 \\ 2 \\ 2 \end{smallmatrix} \text{ and } P_2 = \begin{smallmatrix} e_2 \\ \beta \\ \beta^2 \end{smallmatrix} \simeq \begin{smallmatrix} 2 \\ 2 \\ 2 \end{smallmatrix}.$$

We show that $\mathcal{H}(\mathsf{2\text{-}silt}\,\Lambda_4)$ is of type $\mathcal{H}_{1,5}$ as follows,

$$\begin{tikzcd}
\begin{bmatrix} 0\to P_1 \\ \oplus \\ 0\to P_2 \end{bmatrix} \arrow[rrrrr] \arrow[d] & & & & & \begin{bmatrix} P_1\to 0 \\ \oplus \\ 0\to P_2 \end{bmatrix} \arrow[d] \\
\begin{bmatrix} 0\to P_1 \\ \oplus \\ P_2\xrightarrow{f_1} P_1^{\oplus 3} \end{bmatrix} \arrow[r] & \begin{bmatrix} P_2\xrightarrow{f_2} P_1^{\oplus 2} \\ \oplus \\ P_2\xrightarrow{f_1} P_1^{\oplus 3} \end{bmatrix} \arrow[r] & \begin{bmatrix} P_2\xrightarrow{f_2} P_1^{\oplus 2} \\ \oplus \\ P_2^{\oplus 2}\xrightarrow{f_3} P_1^{\oplus 3} \end{bmatrix} \arrow[r] & \begin{bmatrix} P_2\xrightarrow{\mu} P_1 \\ \oplus \\ P_2^{\oplus 2}\xrightarrow{f_3} P_1^{\oplus 3} \end{bmatrix} \arrow[r] & \begin{bmatrix} P_2\xrightarrow{\mu} P_1 \\ \oplus \\ P_2\to 0 \end{bmatrix} \arrow[r] & \begin{bmatrix} P_1\to 0 \\ \oplus \\ P_2\to 0 \end{bmatrix}
\end{tikzcd}$$

where

$$f_1 = \begin{pmatrix} \mu \\ \mu\beta \\ \mu\beta^2 \end{pmatrix},\ f_2 = \begin{pmatrix} \mu \\ \mu\beta \end{pmatrix},\ f_3 = \begin{pmatrix} \mu & 0 \\ -\mu\beta & \mu \\ 0 & \mu\beta \end{pmatrix}.$$

Since $\mathsf{Hom}_{\Lambda_4}(P_2, P_1) = e_1\Lambda_4 e_2 = K\mu \oplus K\mu\beta \oplus K\mu\beta^2$ and $\mathsf{Hom}_{\Lambda_4}(P_1, P_2) = 0$, it is not difficult to compute the left mutations $\mu^-_{P_1}(\Lambda_4)$ and $\mu^-_{P_2}(\Lambda_4)$. According to the bijection introduced in Theorem 2.2.6, one can find the corresponding two-term silting complexes. We only show details for the rest of the steps.

(1) Let $T_2 = X \oplus Y := (0 \longrightarrow P_1) \oplus (P_2 \xrightarrow{f_1} P_1^{\oplus 3})$. Then, $\mu^-_Y(T_2)$ does not belong to $\mathsf{2\text{-}silt}\ \Lambda_4$ and therefore, we ignore this mutation. To compute $\mu^-_X(T_2)$, we take a triangle

$$X \xrightarrow{\pi} Y \longrightarrow \mathsf{cone}(\pi) \longrightarrow X[1] \text{ with } \pi = \left(0, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right).$$

We may check that π is a minimal left $\mathsf{add}(Y)$-approximation. In fact, by the definition,

- if we compose π with the endomorphism

$$\begin{tikzcd} Y: & P_2 \arrow[r, "f_1"] \arrow[d, "k_1e_2+k_2\beta+k_3\beta^2"'] & P_1^{\oplus 3} \arrow[d, "\begin{pmatrix} k_1 & k_2 & k_3 \\ 0 & k_1 & k_2 \\ 0 & 0 & k_1 \end{pmatrix}"] \\ Y: & P_2 \arrow[r, "f_1"] & P_1^{\oplus 3} \end{tikzcd} \text{, where } k_1, k_2, k_3 \in K,$$

  then all elements of $\mathsf{Hom}_{\mathsf{K^b(proj}\ \Lambda_4)}(X, Y)$ are obtained;
- if $\begin{pmatrix} k_1 & k_2 & k_3 \\ 0 & k_1 & k_2 \\ 0 & 0 & k_1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, then $k_1 = 1$ and $k_2 = k_3 = 0$.

Hence, π is indeed a minimal left $\mathsf{add}(Y)$-approximation. We apply Lemma 2.2.8 by setting $X = Z = M = P_1$, $Y = P_2$ and $X \oplus Y \to Z \oplus X \oplus M$ by

$$\begin{pmatrix} 0 & \mu \\ 1 & \mu\beta^2 \\ 0 & \mu\beta \end{pmatrix}.$$

Then, we have

$$\mathsf{cone}(\pi) = (\ P_1 \oplus P_2 \xrightarrow{\begin{pmatrix} 0 & \mu \\ 0 & \mu\beta \\ 1 & \mu\beta^2 \end{pmatrix}} P_1^{\oplus 3}\ ) \sim_h (P_2 \xrightarrow{f_2} P_1^{\oplus 2}).$$

Thus, $\mu^-_X(T_2) = (P_2 \xrightarrow{f_2} P_1^{\oplus 2}) \oplus (P_2 \xrightarrow{f_1} P_1^{\oplus 3})$.

(2) Let $T_{21} = X \oplus Y := (P_2 \xrightarrow{f_2} P_1^{\oplus 2}) \oplus (P_2 \xrightarrow{f_1} P_1^{\oplus 3})$. Then, $\mu^-_X(T_{21}) \notin \mathsf{2\text{-}silt}\ \Lambda_4$. To compute $\mu^-_Y(T_{21})$, we take a triangle

$$Y \xrightarrow{\pi} X^{\oplus 3} \longrightarrow \mathsf{cone}(\pi) \longrightarrow Y[1] \text{ with } \pi = \left(\begin{pmatrix} e_2 \\ \beta \\ \beta^2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right).$$

Then, π is a minimal left $\mathsf{add}(X)$-approximation. (In fact, we have $\mathsf{End}_{\mathsf{K^b(proj}\ \Lambda_4)}(X) = K$ since $\mathsf{End}_{\Lambda_4}(P_1) = K$. Then,

$$\mathsf{End}_{\mathsf{K^b(proj}\ \Lambda_4)}(X^{\oplus 3}) \simeq \mathrm{Mat}(3, 3, K).$$

Secondly, $\lambda \circ \pi = \pi$ for $\lambda \in \mathrm{Mat}(3, 3, K)$ implies that λ is the identity. Thus, π is indeed a minimal left $\mathsf{add}(X)$-approximation.) Similar to the above, we can apply Lemma 2.2.8 twice by precise settings. Then, we have

$$\mathsf{cone}(\pi) = (\ P_2 \xrightarrow{\begin{pmatrix} -\mu \\ -\mu\beta \\ -\mu\beta^2 \\ e_2 \\ \beta \\ \beta^2 \end{pmatrix}} P_1^{\oplus 3} \oplus P_2^{\oplus 3} \xrightarrow{\begin{pmatrix} 1&0&0&\mu&0&0 \\ 0&1&0&\mu\beta&0&0 \\ 0&1&0&0&\mu&0 \\ 0&0&1&0&\mu\beta&0 \\ 0&0&1&0&0&\mu \\ 0&0&0&0&0&\mu\beta \end{pmatrix}} P_1^{\oplus 6}\ ) \sim_h (P_2^{\oplus 2} \xrightarrow{f_3} P_1^{\oplus 3}).$$

Thus, $\mu_Y^-(T_{21}) = (P_2 \xrightarrow{f_2} P_1^{\oplus 2}) \oplus (P_2^{\oplus 2} \xrightarrow{f_3} P_1^{\oplus 3})$.

(3) Let $T_{212} = X \oplus Y := (P_2 \xrightarrow{f_2} P_1^{\oplus 2}) \oplus (P_2^{\oplus 2} \xrightarrow{f_3} P_1^{\oplus 3})$. Then, $\mu_Y^-(T_{212}) \notin \mathsf{2\text{-}silt}\ \Lambda_4$. To compute $\mu_X^-(T_{212})$, we take a triangle

$$X \xrightarrow{\pi} Y \longrightarrow \mathsf{cone}(\pi) \longrightarrow X[1] \ \text{ with } \pi = \left(\left(\begin{smallmatrix} 0 \\ e_2 \end{smallmatrix} \right), \left(\begin{smallmatrix} 0&0 \\ 1&0 \\ 0&1 \end{smallmatrix} \right) \right).$$

Then, π is a minimal left $\mathsf{add}(Y)$-approximation. In fact, if we compose π with

$$\begin{tikzcd} Y: & P_2^{\oplus 2} \arrow[r, "f_3"] \arrow[d, "\left(\begin{smallmatrix} k_1e_2 - k_2\beta & k_2e_2 \\ -k_2\beta^2 & k_1e_2 \end{smallmatrix}\right)"'] & P_1^{\oplus 3} \arrow[d, "\left(\begin{smallmatrix} k_1 & k_2 & 0 \\ 0 & k_1 & -k_2 \\ 0 & 0 & k_1 \end{smallmatrix}\right)"] \\ Y: & P_2^{\oplus 2} \arrow[r, "f_3"] & P_1^{\oplus 3} \end{tikzcd} \quad , \text{ where } \ k_1, k_2 \in K.$$

then all elements of $\mathsf{Hom}_{\mathsf{K^b(proj}\ \Lambda_4)}(X, Y)$ are obtained; if $\left(\begin{smallmatrix} k_1 & k_2 & 0 \\ 0 & k_1 & -k_2 \\ 0 & 0 & k_1 \end{smallmatrix} \right) \left(\begin{smallmatrix} 0&0 \\ 1&0 \\ 0&1 \end{smallmatrix} \right) = \left(\begin{smallmatrix} 0&0 \\ 1&0 \\ 0&1 \end{smallmatrix} \right)$, then $k_1 = 1$ and $k_2 = 0$. By applying Lemma 2.2.8 twice, we have

$$\mathsf{cone}(\pi) = (\ P_2 \xrightarrow{\begin{pmatrix} -\mu \\ -\mu\beta \\ 0 \\ e_2 \end{pmatrix}} P_1^{\oplus 2} \oplus P_2^{\oplus 2} \xrightarrow{\begin{pmatrix} 0&0&\mu&0 \\ 1&0&-\mu\beta&\mu \\ 0&1&0&\mu\beta \end{pmatrix}} P_1^{\oplus 3}\ ) \sim_h (P_2 \xrightarrow{\mu} P_1).$$

Thus, $\mu_X^-(T_{212}) = (P_2 \xrightarrow{\mu} P_1) \oplus (P_2^{\oplus 2} \xrightarrow{f_3} P_1^{\oplus 3})$.

(4) Let $T_{2121} = X \oplus Y := (P_2 \xrightarrow{\mu} P_1) \oplus (P_2^{\oplus 2} \xrightarrow{f_3} P_1^{\oplus 3})$. Then, $\mu_X^-(T_{2121}) \notin \mathsf{2\text{-}silt}\ \Lambda_4$. To compute $\mu_Y^-(T_{2121})$, we take a triangle

$$Y \xrightarrow{\pi} X^{\oplus 3} \longrightarrow \mathsf{cone}(\pi) \longrightarrow Y[1] \ \text{ with } \pi = \left(\left(\begin{smallmatrix} e_2 & 0 \\ -\beta & e_2 \\ 0 & \beta \end{smallmatrix} \right), \left(\begin{smallmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{smallmatrix} \right) \right).$$

Then, π is a minimal left $\mathsf{add}(X)$-approximation since $\mathsf{End}_{\mathsf{K^b(proj}\ \Lambda_4)}(X) = K$. Then,

$$\mathsf{cone}(\pi) = (\ P_2^{\oplus 2} \xrightarrow{\begin{pmatrix} -\mu & 0 \\ \mu\beta & -\mu \\ 0 & -\mu\beta \\ e_2 & 0 \\ -\beta & e_2 \\ 0 & \beta \end{pmatrix}} P_1^{\oplus 3} \oplus P_2^{\oplus 3} \xrightarrow{\begin{pmatrix} 1&0&0&\mu&0&0 \\ 0&1&0&0&\mu&0 \\ 0&0&1&0&0&\mu \end{pmatrix}} P_1^{\oplus 3}\ ) \sim_h (P_2 \longrightarrow 0).$$

Thus, $\mu_Y^-(T_{2121}) = (P_2 \xrightarrow{\mu} P_1) \oplus (P_2 \longrightarrow 0)$.

(5) Let $T_{21212} = X \oplus Y := (P_2 \xrightarrow{\mu} P_1) \oplus (P_2 \longrightarrow 0)$. Then, it is clear that $\mu_Y^-(T_{21212})$ does not belong to $\mathsf{2\text{-}silt}\ \Lambda_4$ and $\mu_X^-(T_{21212}) = (P_1 \longrightarrow 0) \oplus (P_2 \longrightarrow 0)$.

To sum up the above, we deduce that $\mathcal{H}(\mathsf{2\text{-}silt}\ \Lambda_4)$ is of type $\mathcal{H}_{1,5}$. By Theorem 2.2.6, this is equivalent to saying that $\mathcal{H}(\mathsf{s\tau\text{-}tilt}\ \Lambda_4)$ is of type $\mathcal{H}_{1,5}$. $\square$

We point out that the Hasse quiver $\mathcal{H}(\mathsf{s\tau\text{-}tilt}\ \Lambda_3)$ is of type $\mathcal{H}_{1,3}$ as follows,

$$\begin{tikzcd} \begin{smallmatrix} 1 \\ 2 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \arrow[rrr] \arrow[d] & & & \begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \arrow[d] \\ \begin{smallmatrix} 1 \\ 2 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} & 1 \\ 1 & 2 \\ 2 & \end{smallmatrix} \arrow[r] & 1 \oplus \begin{smallmatrix} & 1 \\ 1 & 2 \\ 2 & \end{smallmatrix} \arrow[r] & 1 \arrow[r] & 0 \end{tikzcd} .$$

Lemma 3.1.3. *The two-point algebra Λ_5 is minimal τ-tilting infinite.*

Proof. Note that Λ_5 is a gentle algebra and it is representation-infinite by Hoshino and Miyachi's result [HM, Theorem A]. Besides, Plamondon [Pl, Theorem 1.1] showed that a gentle algebra is τ-tilting finite if and only if it is representation-finite. Therefore, Λ_5 is τ-tilting infinite. For the minimality, we may consider

$$\hat{\Lambda}_5 := \Lambda_5 / < \alpha\mu\beta >$$

since the socle of Λ_5 is $K\alpha\mu\beta \oplus K\mu\beta \oplus K\beta$ and any proper quotient Λ_5/I of Λ_5 satisfies $\alpha\mu\beta \in I$. We denote by P_1 and P_2 the indecomposable projective $\hat{\Lambda}_5$-modules, then

$$P_1 = \begin{smallmatrix} & e_1 & \\ \alpha & & \mu \\ \alpha\mu & & \mu\beta \end{smallmatrix} \simeq \begin{smallmatrix} & 1 & \\ 1 & & 2 \\ 2 & & 2 \end{smallmatrix} \text{ and } P_2 = \begin{smallmatrix} e_2 \\ \beta \end{smallmatrix} \simeq \begin{smallmatrix} 2 \\ 2 \end{smallmatrix}.$$

We calculate the left mutation sequences starting from $P_1 \oplus P_2$ and ending at 0, so that $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ \hat{\Lambda}_5)$ is of type $\mathcal{H}_{1,4}$ as follows,

$$\begin{tikzcd}
\begin{smallmatrix} & 1 & \\ 1 & & 2 \\ 2 & & 2 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \arrow[rrrr] \arrow[d] & & & & \begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \arrow[d] \\
\begin{smallmatrix} & 1 & \\ 1 & & 2 \\ 2 & & 2 \end{smallmatrix} \oplus \begin{smallmatrix} & & & & 1 & \\ & 1 & & 2 & & 1 \\ 1 & & 2 & & & 2 \\ 2 & & & & & \end{smallmatrix} \arrow[r] & \begin{smallmatrix} 1 \\ 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} & & & & 1 & \\ & 1 & & 2 & & 1 \\ 1 & & 2 & & & 2 \\ 2 & & & & & \end{smallmatrix} \arrow[r] & \begin{smallmatrix} 1 \\ 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \arrow[r] & \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \arrow[r] & 0
\end{tikzcd}.$$

This implies that $\hat{\Lambda}_5$ is τ-tilting finite and Λ_5 is minimal τ-tilting infinite. $\square$

Lemma 3.1.4. *The two-point algebras Λ_7, Λ_8, Λ_9 and Λ_{10} are τ-tilting finite.*

Proof. Since Λ_7, Λ_8 and Λ_9 are quotient algebras of Λ_{10}, it suffices to show that Λ_{10} is τ-tilting finite. The indecomposable projective modules of Λ_{10} are

$$P_1 = e_1\Lambda_{10} = \begin{smallmatrix} & & & e_1 \\ & & \alpha & & \mu \\ & \alpha^2 & & \alpha\mu & \\ \alpha^2\mu & & & & \end{smallmatrix} \simeq \begin{smallmatrix} & & 1 \\ & 1 & 2 \\ 1 & 2 & \\ 2 & & \end{smallmatrix} \text{ and } P_2 = e_2\Lambda_{10} = \begin{smallmatrix} e_2 \\ \nu \\ \nu\alpha \\ \nu\alpha^2 \end{smallmatrix} \simeq \begin{smallmatrix} 2 \\ 1 \\ 1 \\ 1 \end{smallmatrix}.$$

Since $\mathsf{Hom}_{\Lambda_{10}}(P_1, P_2) = e_2\Lambda_{10}e_1 = K\nu \oplus K\nu\alpha \oplus K\nu\alpha^2$ (resp., $\mathsf{Hom}_{\Lambda_{10}}(P_2, P_1) = e_1\Lambda_{10}e_2 = K\mu \oplus K\alpha\mu \oplus K\alpha^2\mu$), we know that the computation of the left mutation sequence started at P_1 (resp., P_2) is similar to that of Λ_4 (resp., Λ_4^{op}). Then, by Proposition 2.1.8 and the calculation in Lemma 3.1.2, we deduce that the Hasse quiver $\mathcal{H}(\mathsf{2\text{-}silt}\ \Lambda_{10})$ is as follows,

$$\begin{tikzcd}
\begin{bmatrix} 0 \to P_1 \\ \oplus \\ 0 \to P_2 \end{bmatrix} \arrow[r] \arrow[d] & \begin{bmatrix} P_1 \xrightarrow{g_1} P_2^{\oplus 3} \\ \oplus \\ 0 \to P_2 \end{bmatrix} \arrow[r] & \begin{bmatrix} P_1 \xrightarrow{g_1} P_2^{\oplus 3} \\ \oplus \\ P_1 \xrightarrow{g_2} P_2^{\oplus 2} \end{bmatrix} \arrow[r] & \begin{bmatrix} P_1^{\oplus 2} \xrightarrow{g_3} P_2^{\oplus 3} \\ \oplus \\ P_1 \xrightarrow{g_2} P_2^{\oplus 2} \end{bmatrix} \arrow[r] & \begin{bmatrix} P_1^{\oplus 2} \xrightarrow{g_3} P_2^{\oplus 3} \\ \oplus \\ P_1 \xrightarrow{\nu} P_2 \end{bmatrix} \arrow[d] \\
\begin{bmatrix} 0 \to P_1 \\ \oplus \\ P_2 \xrightarrow{\mu} P_1 \end{bmatrix} \arrow[d] & & & & \begin{bmatrix} P_1 \to 0 \\ \oplus \\ P_1 \xrightarrow{\nu} P_2 \end{bmatrix} \arrow[d] \\
\begin{bmatrix} P_2^{\oplus 3} \xrightarrow{f_1} P_1^{\oplus 2} \\ \oplus \\ P_2 \xrightarrow{\mu} P_1 \end{bmatrix} \arrow[r] & \begin{bmatrix} P_2^{\oplus 3} \xrightarrow{f_1} P_1^{\oplus 2} \\ \oplus \\ P_2^{\oplus 2} \xrightarrow{f_2} P_1 \end{bmatrix} \arrow[r] & \begin{bmatrix} P_2^{\oplus 3} \xrightarrow{f_3} P_1 \\ \oplus \\ P_2^{\oplus 2} \xrightarrow{f_2} P_1 \end{bmatrix} \arrow[r] & \begin{bmatrix} P_2^{\oplus 3} \xrightarrow{f_3} P_1 \\ \oplus \\ P_2 \to 0 \end{bmatrix} \arrow[r] & \begin{bmatrix} P_1 \to 0 \\ \oplus \\ P_2 \to 0 \end{bmatrix}
\end{tikzcd},$$

where $f_1 = \left(\begin{smallmatrix} \alpha\mu & \mu & 0 \\ 0 & -\alpha\mu & \mu \end{smallmatrix}\right)$, $f_2 = \left(\begin{smallmatrix} \alpha\mu & \mu \end{smallmatrix}\right)$, $f_3 = \left(\begin{smallmatrix} \alpha^2\mu & \alpha\mu & \mu \end{smallmatrix}\right)$ and

$$g_1 = \begin{pmatrix} \nu \\ \nu\alpha \\ \nu\alpha^2 \end{pmatrix},\ g_2 = \left(\begin{smallmatrix} \nu \\ \nu\alpha \end{smallmatrix}\right),\ g_3 = \begin{pmatrix} \nu & 0 \\ -\nu\alpha & \nu \\ 0 & \nu\alpha \end{pmatrix}.$$

We conclude that $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ \Lambda_{10}) \simeq \mathcal{H}(2\text{-}\mathsf{silt}\ \Lambda_{10})$ is of type $\mathcal{H}_{5,5}$. Thus, Λ_7, Λ_8, Λ_9 and Λ_{10} are τ-tilting finite. Next, we determine the type of $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ \Lambda_i)$ for $i = 7, 8, 9$.

(1) The indecomposable projective Λ_7-modules are

$$P_1 = \begin{smallmatrix} & e_1 & \\ \alpha & & \mu \\ \alpha\mu & & \end{smallmatrix} \text{ and } P_2 = \begin{smallmatrix} e_2 \\ \nu \end{smallmatrix}.$$

We give the Hasse quiver $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ \Lambda_7)$ by direct calculation as follows,

$$\begin{tikzcd}
\begin{smallmatrix} & 1 & \\ 1 & & 2 \\ 2 & & \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \arrow[r] \arrow[d] & 2 \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \arrow[rr] & & 2 \arrow[d] \\
\begin{smallmatrix} & 1 & \\ 1 & & 2 \\ 2 & & \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 1 \\ 2 \end{smallmatrix} \arrow[r] & \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 1 \\ 2 \end{smallmatrix} \arrow[r] & \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \arrow[r] & 0
\end{tikzcd}.$$

Then, $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ \Lambda_7)$ is of type $\mathcal{H}_{2,3}$.

(2) Similarly, the Hasse quiver $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ \Lambda_8)$ is given as follows,

$$\begin{tikzcd}
\begin{smallmatrix} & 1 & \\ 1 & & 2 \\ 2 & & \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 1 \end{smallmatrix} \arrow[r] \arrow[d] & \begin{smallmatrix} 2 & & \\ 1 & & 2 \\ & 1 & \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 1 \end{smallmatrix} \arrow[r] & \begin{smallmatrix} 2 & & \\ 1 & & 2 \\ & 1 & \end{smallmatrix} \oplus 2 \arrow[r] & 2 \arrow[d] \\
\begin{smallmatrix} & 1 & \\ 1 & & 2 \\ 2 & & \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 1 \\ 2 \end{smallmatrix} \arrow[r] & \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 1 \\ 2 \end{smallmatrix} \arrow[r] & \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \arrow[r] & 0
\end{tikzcd}.$$

Then, $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ \Lambda_8)$ is of type $\mathcal{H}_{3,3}$.

(3) Let Q_1 and Q_2 be the indecomposable projective Λ_9-modules. Then,

$$Q_1 = e_1\Lambda_9 = \begin{smallmatrix} & & e_1 \\ & \alpha & \mu \\ \alpha^2 & \alpha\mu & \\ \alpha^2\mu & & \end{smallmatrix} \simeq \begin{smallmatrix} & & 1 \\ & 1 & 2 \\ 1 & 2 & \\ 2 & & \end{smallmatrix} \text{ and } Q_2 = e_2\Lambda_9 = \begin{smallmatrix} e_2 \\ \nu \end{smallmatrix} \simeq \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}.$$

Since $\mathsf{Hom}_{\Lambda_9}(Q_1, Q_2) = e_2\Lambda_9 e_1 = K\nu$ and $\mathsf{Hom}_{\Lambda_9}(Q_2, Q_1) = e_1\Lambda_9 e_2 = K\mu \oplus K\alpha\mu \oplus K\alpha^2\mu$, the computation of the left mutation sequence started at Q_2 is similar to that of Λ_4^{op}. Then, by Proposition 2.1.8 and the calculation in Lemma 3.1.2, we deduce that $\mathcal{H}(2\text{-}\mathsf{silt}\ \Lambda_9)$ is presented as follows,

$$\begin{tikzcd}[column sep=small]
\begin{bmatrix} 0 \to Q_1 \\ \oplus \\ 0 \to Q_2 \end{bmatrix} \arrow[rr] \arrow[d] & & \begin{bmatrix} Q_1 \xrightarrow{\nu} Q_2 \\ \oplus \\ 0 \to Q_2 \end{bmatrix} \arrow[rr] & & \begin{bmatrix} Q_1 \to 0 \\ \oplus \\ Q_1 \xrightarrow{\nu} Q_2 \end{bmatrix} \arrow[dd] \\
\begin{bmatrix} 0 \to Q_1 \\ \oplus \\ Q_2 \xrightarrow{\mu} Q_1 \end{bmatrix} \arrow[d] & & & & \\
\begin{bmatrix} Q_2^{\oplus 3} \xrightarrow{f_1} Q_1^{\oplus 2} \\ \oplus \\ Q_2 \xrightarrow{\mu} Q_1 \end{bmatrix} \arrow[r] & \begin{bmatrix} Q_2^{\oplus 3} \xrightarrow{f_1} Q_1^{\oplus 2} \\ \oplus \\ Q_2^{\oplus 2} \xrightarrow{f_2} Q_1 \end{bmatrix} \arrow[r] & \begin{bmatrix} Q_2^{\oplus 3} \xrightarrow{f_3} Q_1 \\ \oplus \\ Q_2^{\oplus 2} \xrightarrow{f_2} Q_1 \end{bmatrix} \arrow[r] & \begin{bmatrix} Q_2^{\oplus 3} \xrightarrow{f_3} Q_1 \\ \oplus \\ Q_2 \to 0 \end{bmatrix} \arrow[r] & \begin{bmatrix} Q_1 \to 0 \\ \oplus \\ Q_2 \to 0 \end{bmatrix}
\end{tikzcd},$$

where $f_1 = \left(\begin{smallmatrix} \alpha\mu & \mu & 0 \\ 0 & -\alpha\mu & \mu \end{smallmatrix}\right)$, $f_2 = \left(\begin{smallmatrix} \alpha\mu & \mu \end{smallmatrix}\right)$ and $f_3 = \left(\begin{smallmatrix} \alpha^2\mu & \alpha\mu & \mu \end{smallmatrix}\right)$. By Theorem 2.2.6, we conclude that $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ \Lambda_9) \simeq \mathcal{H}(2\text{-}\mathsf{silt}\ \Lambda_9)$ is of type $\mathcal{H}_{2,5}$. $\square$

Lemma 3.1.5. *The two-point algebras Λ_{11} and Λ_{12} are τ-tilting finite.*

Proof. We calculate the Hasse quiver $\mathcal{H}(\mathsf{s\tau\text{-}tilt}\ \Lambda_{11})$ directly as follows,

$$\begin{tikzcd}
\begin{smallmatrix} &1& \\ 1&&2 \\ &&2 \end{smallmatrix} \oplus \begin{smallmatrix} &2& \\ 2&&1 \\ &&1 \end{smallmatrix} \arrow[r] \arrow[d] & \begin{smallmatrix} &&&2& \\ &2&1&&2 \\ 2&1&&& \end{smallmatrix} \oplus \begin{smallmatrix} &2& \\ 2&&1 \\ &&1 \end{smallmatrix} \arrow[r] & \begin{smallmatrix} &&&2& \\ &2&1&&2 \\ 2&1&&& \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \arrow[r] & \begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \arrow[d] \\
\begin{smallmatrix} &1& \\ 1&&2 \\ &&2 \end{smallmatrix} \oplus \begin{smallmatrix} &&&1& \\ &1&2&&1 \\ 1&2&&& \end{smallmatrix} \arrow[r] & \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} &&&1& \\ &1&2&&1 \\ 1&2&&& \end{smallmatrix} \arrow[r] & \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \arrow[r] & 0
\end{tikzcd}.$$

Similarly, the Hasse quiver $\mathcal{H}(\mathsf{s\tau\text{-}tilt}\ \Lambda_{12})$ is shown as follows,

$$\begin{tikzcd}
\begin{smallmatrix} &1& \\ 1&&2 \\ 2&&2 \end{smallmatrix} \oplus \begin{smallmatrix} &2& \\ 1&&2 \end{smallmatrix} \arrow[r] \arrow[d] & \begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} &2& \\ 1&&2 \end{smallmatrix} \arrow[rrr] & & & \begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \arrow[d] \\
\begin{smallmatrix} &1& \\ 1&&2 \\ 2&&2 \end{smallmatrix} \oplus \begin{smallmatrix} &&&1& \\ &1&2&&1 \\ 1&&&&2 \\ 2&&&& \end{smallmatrix} \arrow[r] & \begin{smallmatrix} 1 \\ 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} &&&1& \\ &1&2&&1 \\ 1&&&&2 \\ 2&&&& \end{smallmatrix} \arrow[r] & \begin{smallmatrix} 1 \\ 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \arrow[r] & \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \arrow[r] & 0
\end{tikzcd}.$$

Then, the statement follows from Proposition 2.1.6. □

We summarize the results in this section as follows. We know that Λ_2 and Λ_5 are minimal τ-tilting infinite. For others, we have

Λ_i	Λ_1	Λ_3	Λ_4	Λ_6	Λ_7	Λ_8	Λ_9	Λ_{10}	Λ_{11}	Λ_{12}
$\#\mathsf{s\tau\text{-}tilt}\ \Lambda_i$	5	6	8	6	7	8	9	12	8	8
Type	$\mathcal{H}_{1,2}$	$\mathcal{H}_{1,3}$	$\mathcal{H}_{1,5}$	$\mathcal{H}_{2,2}$	$\mathcal{H}_{2,3}$	$\mathcal{H}_{3,3}$	$\mathcal{H}_{2,5}$	$\mathcal{H}_{5,5}$	$\mathcal{H}_{3,3}$	$\mathcal{H}_{2,4}$

.

3.2 Minimal wild two-point algebras

In this section, we can solve the question we mentioned at the beginning of this chapter. We first recall the complete classification for the representation type of two-point algebras. A complete list of (strongly) minimal wild two-point algebras is given by Han [Han], which is displayed by Table W in his paper. (See also Appendix A.1 of this thesis.)

Proposition 3.2.1 ([Han, Theorem 1])**.** *Let A be a two-point algebra. Up to isomorphism and duality, A is representation-finite or tame if and only if A degenerates to a quotient algebra of an algebra from Table T*[1]*, and A is wild if and only if A has a minimal wild algebra from Table W as a quotient algebra.*

Proposition 3.2.2 ([Han, Theorem 2])**.** *A two-point algebra A is strongly minimal wild if and only if it is one of the algebras $W_2 \sim W_{34}$ in Table W.*

Now, we are able to state our first result in this chapter.

[1] We mention that some relations are omitted in the original Table T in [Han] so that several algebras (e.g., T_4 and T_5) in the original Table T are not finite-dimensional. However, we have added these omitted relations in this thesis so that all algebras in Table T are finite-dimensional.

Theorem 3.2.3. *Let W_i be a minimal wild two-point algebra from Table W. Then,*

(1) *W_1, W_2, W_3 and W_5 are τ-tilting infinite.*

(2) *Others are τ-tilting finite. Moreover, we have*

W_i	W_4	W_6	W_7	W_8	W_9	W_{10}	W_{11}	W_{12}	W_{13}	W_{14}
$\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ W_i$	5	6	8		6		7	5		10
Type	$\mathcal{H}_{1,2}$	$\mathcal{H}_{1,3}$	$\mathcal{H}_{1,5}$		$\mathcal{H}_{1,3}$		$\mathcal{H}_{1,4}$	$\mathcal{H}_{1,2}$		$\mathcal{H}_{3,5}$

W_i	W_{15}	W_{16}	W_{17}	W_{18}	W_{19}	W_{20}	W_{21}	W_{22}	W_{23}	W_{24}
$\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ W_i$	9	8	9	8	7				8	10
Type	$\mathcal{H}_{2,5}$	$\mathcal{H}_{3,3}$	$\mathcal{H}_{2,5}$	$\mathcal{H}_{3,3}$	$\mathcal{H}_{2,3}$				$\mathcal{H}_{3,3}$	$\mathcal{H}_{3,5}$

W_i	W_{25}	W_{26}	W_{27}	W_{28}	W_{29}	W_{30}	W_{31}	W_{32}	W_{33}	W_{34}
$\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ W_i$	7		8				6			
Type	$\mathcal{H}_{2,3}$		$\mathcal{H}_{2,4}$	$\mathcal{H}_{3,3}$		$\mathcal{H}_{2,4}$	$\mathcal{H}_{2,2}$			

,

where the type of $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ W_i)$ is defined in the beginning of this chapter.

Proof. First, one can easily find that W_1, W_2, W_3 and W_5 have Λ_2 as a quotient algebra and therefore, they are τ-tilting infinite. It is also not difficult to find that W_4 is τ-tilting finite and the Hasse quiver $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ W_4)$ is of type $\mathcal{H}_{1,2}$ as follows,

$$\begin{tikzcd}
\begin{smallmatrix}&1&\\1&2&1\end{smallmatrix}\oplus 2 \arrow[rr] \arrow[d] & & 2 \arrow[d] \\
\begin{smallmatrix}&1&\\1&2&1\end{smallmatrix}\oplus \begin{smallmatrix}&1&\\1&&1\end{smallmatrix} \arrow[r] & \begin{smallmatrix}&1&\\1&&1\end{smallmatrix} \arrow[r] & 0
\end{tikzcd}.$$

W_i	I	A
W_6	α^2	Λ_3^{op}
W_7	α^3	Λ_4^{op}
W_8	α	Λ_4
W_9	α, β^2	Λ_3
W_{10}		
W_{11}	β^2	$\hat{\Lambda}_5$
W_{12}	α, β	Λ_1
W_{13}		
W_{15}	$\nu\mu$	Λ_9^{op}
W_{16}	$\alpha^2, \nu\mu$	Λ_8
W_{17}	α^3	Λ_9^{op}
W_{18}	α^2	Λ_8
W_{19}		Λ_7
W_{20}	$\mu\nu + \nu\mu$	

W_i	I	A
W_{21}	$\alpha\mu\nu$	Λ_7
W_{22}	$\alpha^2, \mu\nu$	
W_{23}	$\alpha^2 + \nu\mu, \nu\alpha\mu$	Λ_8
W_{24}	$\mu\nu$	$\widetilde{W}_{14}^{\mathsf{op}}$
W_{25}	α^2, β	Λ_7
W_{26}	$\alpha, \mu\nu$	Λ_7^{op}
W_{27}	$\mu\nu$	$\Lambda_{12}^{\mathsf{op}}$
W_{28}		Λ_{11}
W_{29}		$\Lambda_{11}^{\mathsf{op}}$
W_{30}		Λ_{12}
W_{31}	$\alpha + \beta, \nu\mu$	Λ_6
W_{32}	$\alpha + \beta, \nu\mu, \mu\nu$	
W_{33}	$\alpha + \beta$	
W_{34}	$\alpha + \beta, \mu\nu$	

Second, we show that $W_6 \sim W_{34}$ (except for W_{14}) are τ-tilting finite by determining the type of $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ W_i)$ for $i = 6, 7, \ldots, 34$ ($i \neq 14$). In order to do this, we can apply Proposition 2.3.6 to construct a two-sided ideal I generated by elements in $C(W_i) \cap \mathsf{rad}(W_i)$ such that $\mathsf{s}\tau\text{-}\mathsf{tilt}\ A \simeq \mathsf{s}\tau\text{-}\mathsf{tilt}\ (W_i/I)$. Then, we can find the type of $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ W_i)$ following Table Λ. Here, we compute the center of an algebra by GAP as shown above, see [GAP]. In particular, we point out that although $\Lambda_7 \not\simeq \widetilde{W}_{21} := W_{21}/I$, but $\mathsf{s}\tau\text{-}\mathsf{tilt}\ \Lambda_7 \simeq \mathsf{s}\tau\text{-}\mathsf{tilt}\ \widetilde{W}_{21}$. To see the latter one, one may check that $\nu\mu + \mu\nu \in C(\widetilde{W}_{21})$ and therefore,

$$\mathsf{s}\tau\text{-}\mathsf{tilt}\ \widetilde{W}_{21} \simeq \mathsf{s}\tau\text{-}\mathsf{tilt}\ \left(\widetilde{W}_{21}/ < \mu\nu, \nu\mu >\right) \simeq \mathsf{s}\tau\text{-}\mathsf{tilt}\ \Lambda_7.$$

Last, we look at the case W_{14}. Note that $\nu\alpha\mu \in C(W_{14})$ and the indecomposable projective modules of $\widetilde{W}_{14} := W_{14}/ < \nu\alpha\mu >$ are

$$P_1 = \begin{smallmatrix} & & e_1 & & \\ & \alpha & & \mu & \\ \alpha^2 & & \alpha\mu & & \end{smallmatrix} \text{ and } P_2 = \begin{smallmatrix} e_2 \\ \nu \\ \nu\alpha \\ \nu\alpha^2 \end{smallmatrix}.$$

Then, we find that $\widetilde{W}_{14}$ is a quotient algebra of Λ_{10} by $\alpha^2\mu$. Thus, by similar calculation with Λ_{10} in the proof of Lemma 3.1.4, one can check that $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ \widetilde{W}_{14})$ is of type $\mathcal{H}_{3,5}$. By Proposition 2.3.6, we deduce that $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ W_{14}) \simeq \mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ \widetilde{W}_{14})$ is of type $\mathcal{H}_{3,5}$. □

Consequently, we can finish the first step toward the complete classification of τ-tilting finite two-point algebras.

Corollary 3.2.4. *A strongly minimal wild two-point algebra A is τ-tilting finite if and only if it does not contain the Kronecker algebra Λ_2 as a quotient algebra.*

Similarly, we have the following result for algebras in Table T.

Theorem 3.2.5. *Let T_i be an algebra from Table T.*

(1) T_1, T_3 and T_{17} are τ-tilting infinite.

(2) Others are τ-tilting finite. Moreover, we have the following posets,

T_i	T_2	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}
$\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ T_i$	*6*		*5*	*6*	*5*		*8*	*12*	*8*
Type	$\mathcal{H}_{1,3}$		$\mathcal{H}_{1,2}$	$\mathcal{H}_{1,3}$	$\mathcal{H}_{1,2}$		$\mathcal{H}_{3,3}$	$\mathcal{H}_{5,5}$	$\mathcal{H}_{3,3}$

T_i	T_{12}	T_{13}	T_{14}	T_{15}	T_{16}	T_{18}	T_{19}	T_{20}	T_{21}
$\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ T_i$	*7*	*6*	*8*	*7*	*9*	*8*	*6*	*7*	*6*
Type	$\mathcal{H}_{2,3}$	$\mathcal{H}_{2,2}$	$\mathcal{H}_{3,3}$	$\mathcal{H}_{2,3}$	$\mathcal{H}_{2,5}$	$\mathcal{H}_{3,3}$	$\mathcal{H}_{2,2}$	$\mathcal{H}_{2,3}$	$\mathcal{H}_{2,2}$

Proof. Similar to the proof of Theorem 3.2.3, one can check that T_1 has Λ_2 as a quotient algebra, T_3 and T_{17} have Λ_5 as a quotient algebra. Hence, T_1, T_3 and T_{17} are τ-tilting infinite. We may also distinguish the following cases.

Case (T_9). Since $\nu\mu, \alpha\mu\nu + \mu\nu\alpha + \nu\alpha\mu \in C(T_9)$ and $\alpha\mu\nu \in C(\widetilde{T}_9)$ with

$$\widetilde{T}_9 := T_9/ < \nu\mu, \nu\alpha\mu, \alpha\mu\nu + \mu\nu\alpha >.$$

Then, we have $\mu\nu \in C(\widetilde{T}_9/ < \alpha\mu\nu >)$ and therefore,

$$\mathsf{s}\tau\text{-}\mathsf{tilt}\ T_9 \simeq \mathsf{s}\tau\text{-}\mathsf{tilt}\ (T_9/ < \mu\nu, \nu\mu, \nu\alpha\mu >) \simeq \mathsf{s}\tau\text{-}\mathsf{tilt}\ \Lambda_8.$$

Case (T_{20}). For any $k \in K/\{0\}$, we have $\mu\nu + \nu\mu \in C(T_{20})$ such that

$$\mathsf{s}\tau\text{-}\mathsf{tilt}\ T_{20} \simeq \mathsf{s}\tau\text{-}\mathsf{tilt}\ \widetilde{T}_{20} \text{ with } \widetilde{T}_{20} := T_{20}/ < \mu\nu, \nu\mu >.$$

Then, the indecomposable projective $\widetilde{T}_{20}$-modules are

$$P_1 = \begin{smallmatrix} & e_1 & \\ \alpha & & \mu \\ & \mu\beta & \end{smallmatrix} \text{ and } P_2 = \begin{smallmatrix} & e_2 & \\ \beta & & \nu \\ & & \nu\alpha \end{smallmatrix},$$

and the Hasse quiver $\mathcal{H}(\mathsf{2\text{-}silt}\ \widetilde{T}_{20})$ is given as follows,

$$\begin{tikzcd}
\begin{smallmatrix} &1& \\ 1&&2 \\ &2& \end{smallmatrix} \oplus \begin{smallmatrix} &2& \\ 2&&1 \\ &&1 \end{smallmatrix} \arrow[r] \arrow[d] & \begin{smallmatrix} &1& \\ 1&&2 \\ &2& \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \arrow[rr] & & \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \arrow[d] \\
\begin{smallmatrix} &&&2& \\ &2&1&&2 \\ 2&1&&& \end{smallmatrix} \oplus \begin{smallmatrix} &2& \\ 2&&1 \\ &&1 \end{smallmatrix} \arrow[r] & \begin{smallmatrix} &&&2& \\ &2&1&&2 \\ 2&1&&& \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \arrow[r] & \begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \arrow[r] & 0
\end{tikzcd}.$$

Thus, $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ T_{20}) \simeq \mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ \widetilde{T}_{20})$ is of type $\mathcal{H}_{2,3}$.

Case (T_{21}). For any $k_1, k_2 \in K/\{0\}$, we have $\mu\nu + \nu\mu \in C(T_{21})$. Similarly, we have $\mathsf{s}\tau\text{-}\mathsf{tilt}\ T_{21} \simeq \mathsf{s}\tau\text{-}\mathsf{tilt}\ (T_{21}/ < \mu\nu, \nu\mu >)$ and the corresponding Hasse quiver is of type $\mathcal{H}_{2,2}$.

For the remaining cases, we may apply Proposition 2.3.6 to construct a two-sided ideal I generated by elements in $C(T_i) \cap \mathsf{rad}(T_i)$ such that $\mathsf{s}\tau\text{-}\mathsf{tilt}\ B \simeq \mathsf{s}\tau\text{-}\mathsf{tilt}\ (T_i/I)$, as follows,

T_i	I	B
T_2	α^2	Λ_3^{op}
T_4	β	
T_5	α, β	Λ_1
T_6	$\alpha + \beta^2$	Λ_3
T_7	$\alpha + \beta$	Λ_1
T_8		
T_{10}	$\nu\alpha^2\mu$	Λ_{10}
T_{11}	$\alpha^2, \nu\alpha\mu$	Λ_8

T_i	I	B
T_{12}	$\alpha^2, \nu\mu$	Λ_7
T_{13}	$\alpha, \mu\nu + \nu\mu$	Λ_6
T_{14}	$\alpha^2 + \nu\mu$	Λ_8
T_{15}	$\alpha^2, \nu\mu$	Λ_7
T_{16}	$\mu\nu$	Λ_9
T_{18}	$\beta, \nu\alpha\mu+$ $\alpha\mu\nu + \mu\nu\alpha$	$\widetilde{T}_9$
T_{19}	$\alpha, \beta, \mu\nu + \nu\mu$	Λ_6

.

□

3.3 Other applications

At the end of this chapter, we give two easy observations. First, we determine the τ-tilting finiteness of two-point symmetric special biserial algebras. We refer to [Sc] for the basic concepts and properties of symmetric special biserial algebras, or equivalently, Brauer graph algebras. In [AIP], the authors classified two-point symmetric special biserial algebras up to Morita equivalence, so that we can determine their τ-tilting finiteness.

Proposition 3.3.1 ([AIP, Theorem 7.1]). *Let A be a two-point symmetric special biserial algebra. Then, A is Morita equivalent to one of $B_i = KQ/I_i$ below, where $m, n, r \in \mathbb{N}$.*

$Q:\ \circ \underset{\nu}{\overset{\mu}{\rightleftarrows}} \circ \qquad I_1 : (\mu\nu)^n\mu = (\nu\mu)^n\nu = 0, n \geqslant 1.$

$Q:\ \alpha \circlearrowright \circ \underset{\nu}{\overset{\mu}{\rightleftarrows}} \circ \qquad$ $I_2 : \alpha\mu = \nu\alpha = 0, \alpha^m = (\mu\nu)^n, m \geqslant 2, n \geqslant 1.$
$I_3 : \alpha^2 = \nu\mu = 0, (\alpha\mu\nu)^n = (\mu\nu\alpha)^n, n \geqslant 1.$

$Q:\ \circ \underset{\nu_1,\nu_2}{\overset{\mu_1,\mu_2}{\rightleftarrows}} \circ$

$I_4 : \mu_1\nu_2 = \nu_2\mu_1 = \mu_2\nu_1 = \nu_1\mu_2 = 0,$
$(\mu_1\nu_1)^m = (\mu_2\nu_2)^n, (\nu_1\mu_1)^m = (\nu_2\mu_2)^n, m, n \geqslant 1.$
$I_5 : \mu_1\nu_2 = \nu_1\mu_1 = \mu_2\nu_1 = \nu_2\mu_2 = 0,$
$(\mu_1\nu_1\mu_2\nu_2)^n = (\mu_2\nu_2\mu_1\nu_1)^n, (\nu_1\mu_2\nu_2\mu_1)^n = (\nu_2\mu_1\nu_1\mu_2)^n, n \geqslant 1.$

$Q:\ \alpha \circlearrowright \circ \underset{\nu}{\overset{\mu}{\rightleftarrows}} \circ \circlearrowleft \beta$

$I_6 : \alpha\mu = \mu\beta = \beta\nu = \nu\alpha = 0,\ \alpha^m = (\mu\nu)^n, \beta^r = (\nu\mu)^n, m, r \geqslant 2, n \geqslant 1.$
$I_7 : \alpha^2 = \nu\mu = \mu\beta = \beta\nu = 0, (\alpha\mu\nu)^n = (\mu\nu\alpha)^n, \beta^m = (\nu\alpha\mu)^n, m \geqslant 2, n \geqslant 1.$
$I_8 : \alpha^2 = \beta^2 = \mu\nu = \nu\mu = 0, (\nu\alpha\mu\beta)^n = (\beta\nu\alpha\mu)^n, (\alpha\mu\beta\nu)^n = (\mu\beta\nu\alpha)^n, n \geqslant 1.$

Then, we have the following observation.

Proposition 3.3.2. *Let B_i be a two-point symmetric special biserial algebra. Then, B_i is τ-tilting finite if $i = 1, 2, 3, 6, 7$; τ-tilting infinite if $i = 4, 5, 8$. Moreover, we have*

B_i	B_1	B_2	B_3	B_6	B_7
$\#\mathsf{s\tau\text{-}tilt}\ B_i$	*6*		*8*	*6*	*8*
Type	$\mathcal{H}_{2,2}$		$\mathcal{H}_{3,3}$	$\mathcal{H}_{2,2}$	$\mathcal{H}_{3,3}$

Proof. One can easily check that B_4 and B_5 have Λ_2 as a quotient algebra, and B_8 has Λ_5 as a quotient algebra. Therefore, B_4, B_5 and B_8 are τ-tilting infinite.

Next, we show the remaining case by case.

Case (B_1). If $n = 1$, then $\mu\nu, \nu\mu \in C(B_1)$. If $n \geqslant 2$, then $\mu\nu + \nu\mu \in C(B_1)$. Both of them satisfy $\mathsf{s\tau\text{-}tilt}\ B_1 \simeq \mathsf{s\tau\text{-}tilt}\ (B_1/ < \mu\nu, \nu\mu >) \simeq \mathsf{s\tau\text{-}tilt}\ \Lambda_6$.

Case (B_2). If $n = 1$, then $\alpha, \nu\mu \in C(B_2)$. If $n \geqslant 2$, then $\alpha, \mu\nu + \nu\mu \in C(B_2)$. Both of them satisfy $\mathsf{s\tau\text{-}tilt}\ B_2 \simeq \mathsf{s\tau\text{-}tilt}\ (B_2/ < \alpha, \mu\nu, \nu\mu >) \simeq \mathsf{s\tau\text{-}tilt}\ \Lambda_6$.

Case (B_3). If $n = 1$, then $\mu\nu, \nu\alpha\mu \in C(B_3)$. If $n \geqslant 2$, then $\alpha\mu\nu + \mu\nu\alpha + \nu\alpha\mu \in C(B_3)$ and $\alpha\mu\nu \in C(\widetilde{B}_3)$ such that $\mu\nu \in C(\widetilde{B}_3/ < \alpha\mu\nu >)$, where

$$\widetilde{B}_3 := B_3/ < \nu\alpha\mu, \alpha\mu\nu + \mu\nu\alpha >.$$

Hence, $\mathsf{s\tau\text{-}tilt}\ B_3 \simeq \mathsf{s\tau\text{-}tilt}\ (B_3/ < \mu\nu, \nu\alpha\mu >) \simeq \mathsf{s\tau\text{-}tilt}\ \Lambda_8$.

Case (B_6). If $n = 1$, then $\alpha, \beta \in C(B_6)$. If $n \geqslant 2$, then $\alpha, \beta, \mu\nu + \nu\mu \in C(B_6)$. Both of them satisfy $\mathsf{s\tau\text{-}tilt}\ B_6 \simeq \mathsf{s\tau\text{-}tilt}\ (B_6/ < \alpha, \beta, \mu\nu, \nu\mu >) \simeq \mathsf{s\tau\text{-}tilt}\ \Lambda_6$.

Case (B_7). If $n=1$, then $\beta, \mu\nu \in C(B_7)$. If $n \geqslant 2$, then $\beta, \alpha\mu\nu + \mu\nu\alpha + \nu\alpha\mu \in C(B_7)$ and $\alpha\mu\nu \in C(\widetilde{B}_7)$ such that $\mu\nu \in C(\widetilde{B}_7/ < \alpha\mu\nu >)$, where

$$\widetilde{B}_7 := B_7/ < \beta, \nu\alpha\mu, \alpha\mu\nu + \mu\nu\alpha >.$$

Thus, $\mathsf{s}\tau\text{-}\mathsf{tilt}\ B_7 \simeq \mathsf{s}\tau\text{-}\mathsf{tilt}\ (B_7/ < \beta, \mu\nu, \nu\alpha\mu >) \simeq \mathsf{s}\tau\text{-}\mathsf{tilt}\ \Lambda_8$. $\square$

Second, we have the following observation.

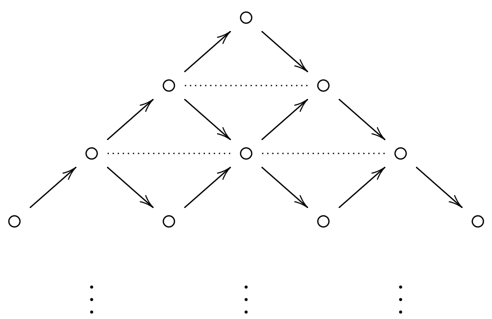
Proposition 3.3.3. *Let A be a connected two-point algebra without loops. Then, A is τ-tilting finite if and only if it is representation-finite.*

Proof. By our assumption, the quiver Q of A does not contain loops. If Q contains multiple arrows, then A has the Kronecker algebra Λ_2 as a quotient algebra and hence, A is τ-tilting infinite. Then, we deduce that if A is τ-tilting finite, then Q is either $\circ \rightleftarrows \circ$ or $\circ \longrightarrow \circ$. On the other hand, any finite-dimensional algebra with quiver $\circ \rightleftarrows \circ$ or $\circ \longrightarrow \circ$ is representation-finite from Bongartz and Gabriel [BoG]. $\square$

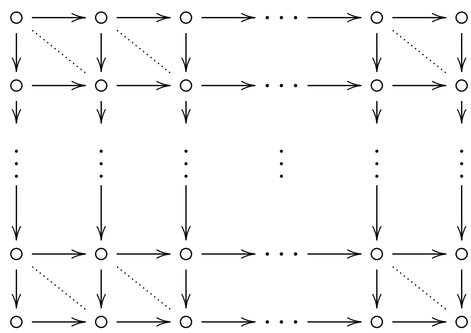
Chapter 4

Simply Connected Algebras

In this chapter, we focus on the class of simply connected algebras, which contains the algebras presented by a triangle quiver or a rectangle quiver with all possible commutativity relations as special cases.



Triangle quiver



Rectangle quiver

The notion of simply connected algebras was first introduced by Bongartz and Gabriel [BoG, Section 6] in representation-finite cases. The importance of these algebras is that we can reduce the representation theory of an arbitrary representation-finite algebra A to that of a representation-finite simply connected algebra B. More precisely, for any representation-finite algebra A, the indecomposable A-modules can be lifted to indecomposable B-modules over a simply connected algebra B, which is contained inside a certain Galois covering of the standard form $\widetilde{A}$ of A, see Proposition 4.1.2 for details.

Soon after, Assem and Skowroński [AS, Section 1.2] introduced the definition for an arbitrary algebra to be simply connected. In the case of representation-finite algebras, this new definition coincides with the definition in [BoG]. So we take this new definition in this thesis (see Definition 4.1.1). Then, the class of simply connected algebras is rather large. For example, it includes tree algebras, tubular algebras, iterated tilted algebras of Euclidean type $\widetilde{\mathbb{D}}_n (n \geqslant 4), \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8$ and so on.

In particular, a subclass of simply connected algebras has been extensively investigated, which is called strongly simply connected algebras and introduced by Skowroński [Sk1]. It is shown in [BrG, Corollary 2.8] that simply connectedness and strongly simply connectedness coincide in the case of representation-finite algebras. Then, the hierarchy (in terms of domestic, polynomial growth and wild) of representation-infinite strongly simply connected algebras has been completely determined, see [B3], [BPS], [NS1], [NS2], [Pe] and [Sk2]. We have reviewed these results in Proposition 4.1.12. We point out that there are some inclusions as follows,

$$\begin{aligned} &\{\text{Simply connected algebras}\} \\ &\supseteq \{\text{Strongly simply connected algebras}\} \\ &\supseteq \{\text{Staircase algebras}\} \cup \{\text{Shifted-staircase algebras}\} \\ &\supseteq \left\{\begin{array}{c}\text{Algebras presented by a triangle or a rectangle} \\ \text{quiver with all possible commutativity relations}\end{array}\right\} \end{aligned}.$$

In the first section, we review the definition of (strongly) simply connected algebras as well as the definitions of critical algebras and Tits forms. In the second section, we show that a simply connected algebra is τ-tilting finite if and only if it is representation-finite. This allows us to determine the τ-tilting finiteness for several classes of algebras, such as tubular algebras, hypercritical algebras and locally hereditary algebras. In particular, we get a complete list of τ-tilting finite sincere simply connected algebras. In the last section, we completely determine the τ-tilting finiteness for algebras presented by a triangle or a rectangle quiver with all possible commutativity relations.

4.1 Basic definitions

Let $A \simeq KQ/I$ be an algebra with $Q = (Q_0, Q_1)$ over an algebraically closed field K. We may regard KQ/I as a K-category (see [BoG, Section 2]) which the class of objects is the set Q_0, and the class of morphisms from i to j is the K-vector space $KQ(i, j)$ of linear combinations of paths in Q with source i and target j, modulo the subspace $I(i, j) := I \cap KQ(i, j)$. We recall some well-known definitions without further reference.

- A is called triangular if Q does not have oriented cycles and loops.
- A is called sincere if there exists an indecomposable A-module M such that all simple A-modules appear in M as composition factors. Otherwise, A is called non-sincere.
- A subcategory B of A is said to be full if for any $i, j \in Q_B$, every morphism $f : i \to j$ in A is also in B; a full subcategory B of A is called convex if any path in Q_A with source and target in Q_B lies entirely in Q_B.
- A relation $\rho = \sum_{i=1}^{n} \lambda_i \omega_i \in I$ with $\lambda_i \neq 0$ is called minimal if $n \geqslant 2$ and for each non-empty proper subset $J \subset \{1, 2, \ldots, n\}$, we have $\sum_{j \in J} \lambda_j \omega_j \notin I$.

We introduce the definition of simply connected algebras as follows. Here, we follow the constructions in [AS, Section 1.2]. Let $A \simeq KQ/I$ be a triangular algebra with a connected quiver $Q = (Q_0, Q_1, s, t)$ and an admissible ideal I. For each arrow $\alpha \in Q_1$, let α^- be its formal inverse with $s(\alpha^-) = t(\alpha)$ and $t(\alpha^-) = s(\alpha)$. Then, we define $Q_1^- := \{\alpha^- \mid \alpha \in Q_1\}$.

A walk is a formal composition $w = w_1w_2\cdots w_n$ with $w_i \in Q_1 \cup Q_1^-$ for all $1 \leqslant i \leqslant n$. We have $s(w) = s(w_1)$, $t(w) = t(w_n)$, $s(w_i) = t(w_{i-1})$ for all $i > 1$ and we denote by 1_x the trivial path at vertex x. For two walks w and u with $s(u) = t(w)$, the composition wu is defined in the obvious way. In particular, $w = 1_{s(w)}w = w1_{t(w)}$. Then, let $\sim$ be the smallest equivalence relation on the set of all walks in Q satisfying the following conditions:

(1) For each $\alpha : x \to y$ in Q_1, we have $\alpha\alpha^- \sim 1_x$ and $\alpha^-\alpha \sim 1_y$.

(2) For each minimal relation $\sum_{i=1}^n \lambda_i\omega_i$ in I, we have $\omega_i \sim \omega_j$ for all $1 \leqslant i, j \leqslant n$.

(3) If u, v, w and w' are walks and $u \sim v$, then we have $wuw' \sim wvw'$ whenever these compositions are defined.

We denote by $[w]$ the equivalence class of a walk w. Clearly, the product wu of two walks w and u induces a product $[w] \cdot [u]$ of $[w]$ and $[u]$. Note that $[wu] = [w] \cdot [u]$.

For a given $x \in Q_0$, the set $\Pi_1(Q, I, x)$ of equivalence classes of all walks w with $s(w) = t(w) = x$ becomes a group via the above product. Since Q is connected, we can always find a walk u from x to y for two different vertices x and y, so that we can define an isomorphism from $\Pi_1(Q, I, x)$ to $\Pi_1(Q, I, y)$ by $[w] \longrightarrow [u]^{-1} \cdot [w] \cdot [u]$. This implies that $\Pi_1(Q, I, x)$ is independent of the choice of x, up to isomorphism. Then, the fundamental group of (Q, I) is defined by

$$\Pi_1(Q, I) := \Pi_1(Q, I, x).$$

Definition 4.1.1 ([AS, Definition 1.2])**.** A triangular algebra A is called simply connected if, for any presentation $A \simeq KQ/I$ as a bound quiver algebra, the fundamental group $\Pi_1(Q, I)$ is trivial.

It follows from [BrG, (1.2)] and [MP, (4.3)] that if A is moreover representation-finite, the above definition coincides with the original definition introduced by Bongartz and Gabriel [BoG, Section 6] that A is simply connected if the Auslander-Reiten quiver of A is simply connected. Let $\Gamma(\mathsf{mod}\ A)$ be the Auslander-Reiten quiver of A, which can be considered as a path category $K\Gamma(\mathsf{mod}\ A)$. For any indecomposable non-projective A-module M, we have the following sequence in $\Gamma(\mathsf{mod}\ A)$,

$$\begin{tikzcd}[row sep=small] & N \arrow[dr, "\alpha_1^-"] & \\ \tau M \arrow[ur, "\alpha_1^+"] \arrow[dr, "\alpha_n^+"'] & \vdots & M \\ & N' \arrow[ur, "\alpha_n^-"'] & \end{tikzcd} .$$

We define $\sigma_M := \sum_{i=1}^n \alpha_i^+\alpha_i^-$ and the mesh-category $m(\Gamma(\mathsf{mod}\ A)) := K\Gamma(\mathsf{mod}\ A)/I_{\Gamma_A}$, which is bounded by the mesh-ideal $I_{\Gamma_A} := \langle \sigma_M \mid \tau M \neq 0 \rangle$. Then, the standard form $\widetilde{A}$ of A (see [BrG, (3.1)]) is defined to be the full subcategory consisting of all projective points of the mesh-category $m(\Gamma(\mathsf{mod}\ A))$. Besides, $\widetilde{A}$ is also representation-finite and $\Gamma(\mathsf{mod}\ \widetilde{A}) = \Gamma(\mathsf{mod}\ A)$, see [BoG, Corollary 5.2] for details. We can see the importance of simply connected algebras in the following proposition.

Proposition 4.1.2 ([BrG, Section 3]). *Let $A = KQ_A/I_A$ be a representation-finite algebra. Then, the standard form $\widetilde{A}$ is Morita equivalent to A and $\widetilde{A}$ admits a Galois covering $F : B \to B/G := \widetilde{A}$, where B is simply connected and G is the fundamental group $\Pi_1(Q_A, I_A)$, which is a finitely generated free group.*

It is worth mentioning that any representation-finite simply connected algebra is standard[1] by [BoG, (6.1)] and therefore, $B = \widetilde{A}$ if A is simply connected in the above.

Example 4.1.3 ([Ass2, Example 2.2]). We give some examples.

(1) All tree algebras are simply connected.
(2) A hereditary algebra KQ is simply connected if and only if Q is a tree.

Next, we look at a subclass of simply connected algebras, which has been studied extensively by quiver and relations.

Definition 4.1.4 ([Sk1, (2.2)]). A triangular algebra A is called strongly simply connected if every convex subcategory of A is simply connected.

We may distinguish the representation-finite cases as follows.

Proposition 4.1.5 ([BrG, Corollary 2.8]). *Let A be a representation-finite triangular algebra. Then, A is simply connected if and only if A is strongly simply connected.*

Example 4.1.6. We have the following examples.

(1) All tree algebras are strongly simply connected.
(2) Completely separating algebras are strongly simply connected, see [Dra].
(3) A hereditary algebra KQ is strongly simply connected if and only if Q is a tree.
(4) Let $A := KQ/I$ with $I :=< \alpha\beta - \gamma\delta, \alpha\lambda - \gamma\mu >$ and the following quiver Q:

$$\begin{tikzcd}
\circ & & \circ \arrow[ll, "\beta"'] \arrow[ddll, "\lambda"] & \\
& & & \circ \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] \\
\circ & & \circ \arrow[ll, "\mu"] \arrow[uull, "\delta"'] &
\end{tikzcd}.$$

Then, A is simply connected but not strongly simply connected, see [Ass2].

We recall the separation property of a triangular algebra $A \simeq KQ/I$, which provides a sufficient condition for A to be simply connected. We denote by P_i the indecomposable projective module at vertex i and $\mathsf{rad}\ P_i$ its radical. Then, P_i is said to have a separated radical (e.g., [ASS, IX, Definition 4.1]) if $\mathsf{rad}\ P_i$ is a direct sum of pairwise non-isomorphic indecomposable modules whose supports are contained in pairwise different connected components of $Q(i)$, where $Q(i)$ is the subquiver of Q obtained by deleting all vertices of Q being a source of a path in Q with target i (including the trivial path from i to i). We say that A satisfies the separation property if every indecomposable projective A-module P has a separated radical.

[1] A modern definition for an algebra A being standard is that A has a universal covering. However, the definition in [BoG] means that the Auslander-Reiten quiver of A is a mesh-category.

Proposition 4.1.7 ([Sk1, (2.3), (4.1)]). *Let A be a triangular algebra.*

(1) If A satisfies the separation property, then it is simply connected.

(2) A is strongly simply connected if and only if every convex subcategory of A (or A^{op}) satisfies the separation property.

Note that the above condition provides a large class of examples of simply connected algebras. We recall that the one-point extension $B = A[M]$ of A by an A-module M is defined by

$$B = A[M] := \begin{bmatrix} A & 0 \\ M & K \end{bmatrix}.$$

If A is simply connected and $M = \mathsf{rad}\ P$ is a separated radical of an indecomposable projective B-module P, then B is also simply connected following [Ass2, Lemma 2.3].

We point out that the characterization of strongly simply connected algebras has been extensively investigated, even though it is not easy to recognize whether a given algebra is simply connected or not.

Proposition 4.1.8 ([AL, Theorem 1.3], [Sk1, (4.1)]). *Let A be a triangular algebra. Then, the following conditions are equivalent.*

(1) A is strongly simply connected.

(2) Every convex subcategory of A (or A^{op}) satisfies the separation property.

(3) There is a presentation (Q, I) of A such that $\Pi_1(Q', I')$ is trivial for any connected full convex bounded subquiver (Q', I') of (Q, I).

4.1.1 Tits form

Let $A \simeq KQ/I$ be a triangular algebra and $\mathbf{N} := \{0, 1, 2, \dots\}$. We recall from [B2, Section 2] that the Tits form $q_A : \mathbb{Z}^{Q_0} \to \mathbb{Z}$ of A is the integral quadratic form defined by

$$q_A(v) = \sum_{i \in Q_0} v_i^2 - \sum_{(i \to j) \in Q_1} v_i v_j + \sum_{i,j \in Q_0} r(i,j) v_i v_j,$$

where $v := (v_i) \in \mathbb{Z}^{Q_0}$ and $r(i,j) = |R \cap I(i,j)|$ with a minimal set $R \subseteq \bigcup_{i,j \in Q_0} I(i,j)$ of generators of the admissible ideal I. Then, the Tits form q_A is called weakly positive if $q_A(v) > 0$ for any $v \neq 0$ in $\mathbf{N}^{Q_0}$, and weakly non-negative if $q_A(v) \geqslant 0$ for any $v \in \mathbf{N}^{Q_0}$.

It is well-known that the Tits form q_A has a close connection with the representation type of A. Here, we recall the related results for (strongly) simply connected algebras.

Proposition 4.1.9 ([SS2, XX, Theorem 2.9, 2.10]). *Let A be a simply connected algebra.*

(1) A is representation-finite if and only if the Tits form q_A is weakly positive, or equivalently, if and only if A does not contain a critical[2] convex subcategory.

(2) If A is strongly simply connected, then A is tame if and only if the Tits form q_A is weakly non-negative, but not weakly positive.

[2]As we mentioned in Remark 4.1.11, the definition of critical algebras used here comes from [SS2, XX, Definition 2.8], but not from [B4].

4.1.2 Critical algebras

Let $\Gamma(\mathsf{mod}\ A)$ be the Auslander-Reiten quiver of A. A connected component C of $\Gamma(\mathsf{mod}\ A)$ is called preprojective if there is no oriented cycle in C, and any module in C is of the form $\tau^{-n}(P)$ for an $n \in \mathbb{N}$ and an indecomposable projective A-module P.

Let $A := K\Delta$ be a hereditary algebra with a tilting A-module T (see Definition 2.0.3). Then, the endomorphism algebra $B := \mathsf{End}_A\ T$ is called a tilted algebra of type Δ. If moreover, T is contained in a preprojective component C of $\Gamma(\mathsf{mod}\ A)$, then we call B a concealed algebra of type Δ.

Definition 4.1.10 ([SS2, XX, Definition 2.8])**.** A critical algebra is one of concealed algebras of Euclidean type $\widetilde{\mathbb{D}}_n (n \geqslant 4)$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ and $\widetilde{\mathbb{E}}_8$.

Remark 4.1.11. The above definition is different from the original definition of critical algebras introduced by Bongartz in [B4]: an algebra A is called critical if A is representation-infinite, but any proper convex subcategory of A is representation-finite. In this thesis, the critical algebras we use are actually the original critical algebras in [B4] obtained by admissible gradings, as described in [B4, Theorem 2].

We point out that critical algebras are strongly simply connected. To show this, we first find in [AS, (1.2)] that critical algebras are simply connected. Since the quiver and relations of critical algebras are given in [B4] (and [HV]), we observe that each proper convex subcategory of a critical algebra is also simply connected and hence, critical algebras are strongly simply connected (see also [SS2, XX, Definition 2.8]).

It is also true that a critical algebra A admits a preprojective component. To show this, one may combine Proposition 4.1.7 and [ASS, IX, Theorem 4.5]. In fact, we observe that any strongly simply connected algebra admits a preprojective component.

We recall from [HV] that tame concealed algebras consist of critical algebras and the concealed algebras of Euclidean type $\widetilde{\mathbb{A}}_n$. Then, tame concealed algebras together with the so-called generalized Kronecker algebras, are precisely the minimal algebras of infinite representation type with a preprojective component. Here, an algebra A is called a minimal algebra of infinite representation type if A is representation-infinite, but A/AeA is representation-finite for any non-zero idempotent e of A.

In the following, we recall some classes of algebras that play an essential role in the representation theory of strongly simply connected algebras. The first class of algebras is tubular algebras, which are introduced by Ringel [Rin, Chapter 5] and have only 6, 8, 9 or 10 simple modules. Tubular algebras are branch-enlargements of the canonical tubular algebras $\mathcal{C}(2,2,2,2)$, $\mathcal{C}(3,3,3)$, $\mathcal{C}(2,4,4)$ and $\mathcal{C}(2,3,6)$, where the canonical algebra $\mathcal{C}(2,2,2,2)$ is defined by the following quiver and relations

$$\begin{tikzcd}
 & \circ \arrow[ddr, "\beta_1"] & \\
 & \circ \arrow[dr, "\beta_2"] & \\
\circ \arrow[uur, "\alpha_1"] \arrow[ur, "\alpha_2"] \arrow[dr, "\alpha_3"] \arrow[ddr, "\alpha_4"] & & \circ \\
 & \circ \arrow[ur, "\beta_3"] & \\
 & \circ \arrow[uur, "\beta_4"] &
\end{tikzcd}
\qquad
\begin{gathered}
\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0,\\
\alpha_1\beta_1 + \lambda\alpha_2\beta_2 + \alpha_4\beta_4 = 0,\\
\lambda \in K/\{0,1\},
\end{gathered}$$

and the canonical algebra $\mathcal{C}(p,q,r)$ with $p \leqslant q \leqslant r$ is defined by the quiver

$$\begin{tikzcd}
 & \circ \arrow[r, "\alpha_2"] & \circ \arrow[r] & \cdots \arrow[r] & \circ \arrow[r, "\alpha_{p-1}"] & \circ \arrow[dr, "\alpha_p"] & \\
\circ \arrow[ur, "\alpha_1"] \arrow[r, "\beta_1"] \arrow[dr, "\gamma_1"'] & \circ \arrow[r, "\beta_2"] & \circ \arrow[r] & \cdots \arrow[r] & \circ \arrow[r, "\beta_{q-1}"] & \circ \arrow[r, "\beta_q"] & \circ \\
 & \circ \arrow[r, "\gamma_2"] & \circ \arrow[r] & \cdots \arrow[r] & \circ \arrow[r, "\gamma_{r-1}"] & \circ \arrow[ur, "\gamma_r"'] &
\end{tikzcd}$$

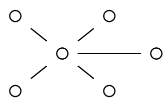
bounded by $\alpha_1\alpha_2\cdots\alpha_p + \beta_1\beta_2\cdots\beta_q + \gamma_1\gamma_2\cdots\gamma_r = 0$. We have known from [HR2, Section 1] that each tubular algebra is derived equivalent to a canonical tubular algebra (of the same type). In particular, [Sk2, Proposition 2.4] implies that every tubular algebra is tame, non-domestic, of polynomial growth.

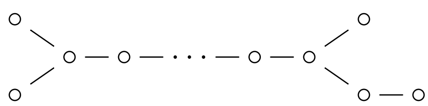
We recall from [Rin, Section 4.1] that an A-module T is called cotilting if it satisfies $|T| = |A|$, $\mathsf{Ext}^1_A(T,T) = 0$ and the injective dimension of T is at most one. Then, A and B are tilting-cotilting equivalent (see [Ha, Corollary 1.7]) if there exists a sequence of algebras $A = A_0, A_1, \dots, A_m = B$ and a sequence of modules $T^i_{A_i} (0 \leqslant i \leqslant m)$, such that $A_{i+1} = \mathsf{End}_{A_i}\, T^i_{A_i}$ and $T^i_{A_i}$ is either a tilting or cotilting module. According to [H, Theorem 1.1], an algebra that is derived equivalent to a tubular algebra is always tilting-cotilting equivalent to one of the canonical tubular algebras. Besides, it is shown in [AS, (1.4)] that an algebra is simply connected if it is tilting-cotilting equivalent to a canonical tubular algebra. Thus, we conclude that tubular algebras are simply connected.

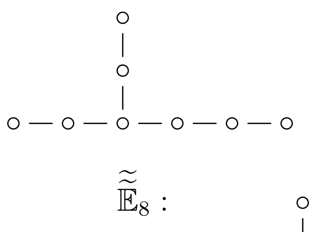
More generally, the algebras which are derived equivalent to tubular algebras are simply connected. However, such an algebra could be representation-finite. One may refer to [Bar, Theorem] for an explicit characterization of representation-finite algebras which are derived equivalent to tubular algebras.

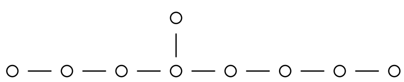
The second class of algebras is the *pg*-critical algebras introduced by Nörenberg and Skowroński [NS2]. These algebras stand for the *p*olynomial *g*rowth critical algebras, that is, representation-infinite tame simply connected algebras which are not of polynomial growth, but every proper convex subcategory is. Following [NS2, Theorem 3.2], one can understand all *pg*-critical algebras by quiver and relations obtained from 31 frames and 3 admissible operations. For the sake of simplicity, we omit these quivers and relations.

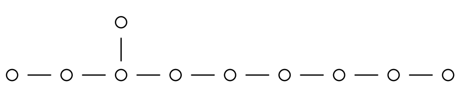
The third class of algebras is the hypercritical algebras introduced by Unger [Un] (see also Lersch [Ler] and Wittman [Wi]). An algebra A is said to be hypercritical if A is a concealed algebra of minimal wild hereditary tree algebras of the following types:

T_5 :


$\widetilde{\widetilde{\mathbb{D}}}_n$:


$\widetilde{\widetilde{\mathbb{E}}}_6$:


$\widetilde{\widetilde{\mathbb{E}}}_7$:


$\widetilde{\widetilde{\mathbb{E}}}_8$:


where in the case of $\widetilde{\widetilde{\mathbb{D}}}_n$ the number of vertices is $n+2$ with $4 \leqslant n \leqslant 8$. Similarly, one can understand hypercritical algebras by quivers and relations ([Un]) and they are strongly simply connected. Actually, they are minimal wild strongly simply connected algebras as shown in Proposition 4.1.12 below.

We point out that tubular algebras and *pg*-critical algebras are not necessarily strongly simply connected while hypercritical algebras must be strongly simply connected.

Proposition 4.1.12 ([BPS, Corollary 1], [Sk2, Theorem 4.1, Corollary 4.3])**.** *Let A be a representation-infinite strongly simply connected algebra.*

(1) A is tame if and only if A does not have a hypercritical algebra as a convex subcategory.

(2) A is tame minimal superpolynomial growth if and only if A is obtained from one of the frames (1)–(16) *in the list of pg-critical algebras in* [NS2] *by admissible operations.*

(3) A is of polynomial growth if and only if A does not have a convex subcategory which is pg-critical or hypercritical.

(4) A is domestic if and only if A does not have a convex subcategory which is tubular or pg-critical or hypercritical.

4.2 Simply connected algebras

In this section, we first show that a τ-tilting finite simply connected algebra is representation-finite. Then, we prove that the τ-tilting finiteness of a non-sincere algebra can be reduced to the τ-tilting finiteness of a sincere algebra. Therefore, we can get a complete list of τ-tilting finite sincere simply connected algebras. Last, we determine the τ-tilting finiteness of several algebras which are related to the representation theory of (strongly) simply connected algebras. We need the following fundamental lemma.

Lemma 4.2.1. *Any critical algebra A is τ-tilting infinite.*

Proof. We have known from Definition 4.1.10 that a critical algebra A is a minimal algebra of infinite representation type with a preprojective component C_A. As we mentioned in the previous section, each pair $(M, \tau M)$ with a non-projective A-module M appears in the Auslander-Reiten quiver $\Gamma(\mathsf{mod}\ A)$ as follows,

$$\begin{tikzcd}[row sep=small] & N \arrow[dr] & \\ \tau M \arrow[ur] \arrow[dr] & \vdots & M\ . \\ & N' \arrow[ur] & \end{tikzcd}$$

Then, by [SS1, X, Proposition 3.2], each pair of indecomposable modules M and N in C_A satisfies $\mathsf{rad}^\infty(M,N) = \mathsf{rad}^\ell(M,N) = 0$ for $\ell \gg 0$. This implies that $\mathsf{Hom}_A(M, \tau M) = 0$ for any $M \in C_A$, i.e., any indecomposable module $M \in C_A$ is a τ-rigid module. Since A is representation-infinite, every connected component of $\Gamma(\mathsf{mod}\ A)$ is infinite (see [ASS, IV, Theorem 5.4] for a proof). Therefore, A has infinitely many pairwise non-isomorphic indecomposable τ-rigid modules and A is τ-tilting infinite by Proposition 2.3.2. $\square$

In the above proof, the following remark is crucial. Some scholars have used this statement, such as Adachi [Ad1] and Mousavand [Mo], to show that τ-tilting finiteness coincides with representation-finiteness for several classes of algebras.

Remark 4.2.2 ([Mo, Remark 2.9])**.** Let A be an algebra with a preprojective component. Then, A is τ-tilting finite if and only if it is representation-finite.

Theorem 4.2.3. *Let A be a simply connected algebra. Then, the following are equivalent.*

(1) A is τ-tilting finite.

(2) A is representation-finite.

(3) A does not have a critical algebra as a convex subcategory.

Proof. By the definition, a convex subcategory B of A is actually a certain idempotent truncation of A. If A is τ-tilting finite, then it cannot have a critical algebra as a convex subcategory by Proposition 2.3.5 and Lemma 4.2.1. This implies (1) $\Rightarrow$ (3). By Proposition 4.1.9, one can find (3) $\Rightarrow$ (2). Last, (2) $\Rightarrow$ (1) is obvious. □

As we mentioned in the first section, we have the following immediate results.

Corollary 4.2.4. *All tubular, pg-critical and hypercritical algebras are τ-tilting infinite.*

Corollary 4.2.5. *Let A be an algebra which is derived equivalent to a tubular algebra. Then, A is τ-tilting finite if and only if it is representation-finite.*

Corollary 4.2.6. *Assume that A is a simply connected algebra. If A is not strongly simply connected, it is τ-tilting infinite.*

Proof. By Proposition 4.1.5, such an algebra A must be representation-infinite. □

Corollary 4.2.7. *Let A be a simply connected algebra and $B = A[M]$ the one-point extension with a separated radical $M = \mathsf{rad}\ P$ for an indecomposable projective B-module P. Then, B is τ-tilting finite if and only if it is representation-finite.*

Proof. This follows from the fact that B is simply connected, see [Ass2, Lemma 2.3]. □

Next, we consider non-sincere and sincere algebras. Let $\{e_1, e_2, \ldots, e_n\}$ be a complete set of pairwise orthogonal primitive idempotents of A. Then, we have

Theorem 4.2.8. *A non-sincere algebra A is τ-tilting finite if and only if A/Ae_iA is τ-tilting finite for any $1 \leqslant i \leqslant n$.*

Proof. If A is τ-tilting finite, then A/Ae_iA is τ-tilting finite following Proposition 2.3.5.

We assume that any A/Ae_iA is τ-tilting finite. Since A is a non-sincere algebra, for any indecomposable A-module M, there exists at least one e_i such that $Me_i = 0$ and we may denote $B_i := A/Ae_iA$. Then, for any indecomposable τ-rigid A-module M, one can always find a suitable i such that M becomes an indecomposable τ-rigid B_i-module. Besides, the number of indecomposable τ-rigid B_i-modules is finite following Proposition 2.3.2. Hence, A is also τ-tilting finite. □

Corollary 4.2.9. *A non-sincere algebra A is τ-tilting finite if and only if all sincere quotient A/AeA is τ-tilting finite for any idempotent e of A.*

Therefore, the study of τ-tilting finiteness for non-sincere algebras reduces to that of sincere algebras. We may apply this strategy to simply connected algebras. Let A be a representation-finite sincere simply connected algebra (it is actually strongly simply connected following Proposition 4.1.5). In [B5], Bongartz introduced a list of 24 infinite families containing all possible A's with $|A| \geqslant 72$. Then, this bound is refined to $|A| \geqslant 14$ in Ringel's book [Rin, Section 6]. Hereafter, Bongartz determined the algebras with $|A| \leqslant 13$ by the graded tree introduced in [BoG]. Finally, Rogat and Tesche [RT] gave a list of all possible A's by Gabriel quiver and relations.

Remark 4.2.10. By Theorem 4.2.3, the list in [RT] provides a complete list of τ-tilting finite sincere (strongly) simply connected algebras.

4.2.1 Some applications

We first consider the class of triangular matrix algebras. We denote by $\mathcal{T}_2(A)$ the algebra of 2×2 upper triangular matrices $\left(\begin{smallmatrix} A & A \\ 0 & A \end{smallmatrix}\right)$ over an algebra A. Then, the category $\mathsf{mod}\ \mathcal{T}_2(A)$ is equivalent to the category whose objects are A-homomorphisms $f: M \to N$ between A-modules M and N, and morphisms are pairs of homomorphisms making the obvious squares commutative. This reminds us that the category $\mathsf{mod}\ \mathcal{T}_2(A)$ is closely connected with the module category of the Auslander algebra of A.

Let A be a representation-finite algebra and $\{M_1, M_2, \ldots, M_s\}$ a complete set of representatives of the isomorphism classes of indecomposable A-modules. Then, the Auslander algebra of A is defined as $\mathsf{End}_A\left(\oplus_{i=1}^s M_i\right)$. We have

Proposition 4.2.11. *Let A be a representation-finite simply connected algebra and $B := \mathsf{End}_A\left(\oplus_{i=1}^s M_i\right)$ its Auslander algebra. Then, the following conditions are equivalent.*

(1) B is τ-tilting finite.

(2) B is representation-finite.

(3) $\mathcal{T}_2(A)$ is τ-tilting finite.

(4) $\mathcal{T}_2(A)$ is representation-finite.

Proof. It follows from [AB, Theorem] that the Auslander algebra of A is simply connected if and only if A is simply connected. Therefore, B is simply connected and (1) $\Leftrightarrow$ (2) follows from Theorem 4.2.3. It is known from [LS1] that $\mathcal{T}_2(A) \simeq \mathcal{T}_2(K) \otimes_K A$. Then, $\mathcal{T}_2(A)$ is simply connected if and only if A is simply connected (see [LS2]). Hence, (3) $\Leftrightarrow$ (4) also follows from Theorem 4.2.3. Lastly, (2) $\Leftrightarrow$ (4) follows from [AR, Theorem 1.1] or [ARS, VI, Proposition 5.8]. $\square$

We point out that if A is a representation-infinite simply connected algebra, $\mathcal{T}_2(A)$ is τ-tilting infinite. In fact, it is easy to check that $\mathcal{T}_2(A)$ has A as a convex subcategory.

Next, we consider the class of iterated tilted algebras. We recall that an algebra A is called iterated tilted of type Δ (see [AH, (1.4)] and [HRS, Theorem 3]) if A is tilting-cotilting equivalent to a path algebra $K\Delta$.

Proposition 4.2.12. *Let A be an iterated tilted algebra of Dynkin type or types $\widetilde{\mathbb{D}}_n, \widetilde{\mathbb{E}}_p, (n \geqslant 4, p = 6, 7$ or $8)$. Then, A is τ-tilting finite if and only if it is representation-finite.*

Proof. It is shown by [Ass1, Proposition 3.5] that iterated tilted algebras of Dynkin type are simply connected, and by [AS, Corollary 1.4] that an iterated tilted algebra of Euclidean type is simply connected if and only if it is of types $\widetilde{\mathbb{D}}_n, \widetilde{\mathbb{E}}_p, (n \geqslant 4, p = 6, 7$ or $8)$. Then, the statement follows from Theorem 4.2.3. □

At the end of this section, we give another application of Theorem 4.2.3. We claim that the result below has been published in a joint work [AHMW] with Takuma Aihara, Takahiro Honma and Kengo Miyamoto.

We recall that an algebra $A \simeq KQ/I$ is called an incidence algebra if Q is the Hasse quiver of a finite poset and I is the ideal generated by all possible commutativity relations, that is, by all elements $w_1 - w_2$ given by the pairs $\{w_1, w_2\}$ of paths in Q having the same source and target. For example, we define

$$Q := \begin{tikzcd}[row sep=small] & \circ \arrow[r, "\alpha_2"] & \circ \arrow[r, "\alpha_3"] & \circ \arrow[dr, "\alpha_4"] & \\ \circ \arrow[ur, "\alpha_1"] \arrow[dr, "\beta_1"'] & & & & \circ \\ & \circ \arrow[r, "\beta_2"] & \circ \arrow[r, "\beta_3"] & \circ \arrow[ur, "\beta_4"'] & \end{tikzcd} \text{ and } I := < \alpha_1\alpha_2\alpha_3\alpha_4 - \beta_1\beta_2\beta_3\beta_4 >.$$

Then, the bound quiver algebra $A := KQ/I$ is an incidence algebra.

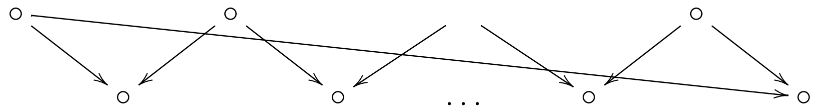
More generally, we recall a larger class of algebras which contains the class of incidence algebras as a special case. We call an algebra A locally hereditary (see [Bau]) if every non-zero homomorphism between indecomposable projective A-modules is a monomorphism. Clearly, hereditary algebras, incidence algebras and tubular algebras are locally hereditary. The class of locally hereditary algebras is rather large and plays an important role in the representation theory of algebras, we refer to [Les] for more details.

Theorem 4.2.13 ([AHMW, Theorem 4.11])**.** *A τ-tilting finite locally hereditary algebra is representation-finite.*

Proof. By the definition, a locally hereditary algebra A has no monomial relations and the quiver Q_A of A is triangular. We observe that if Q_A contains a subquiver of Euclidean type $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ or $\widetilde{\mathbb{E}}_8$, then the corresponding idempotent truncation is a path algebra because $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$ are trees and only possible relations are monomial relations.

We assume that A is τ-tilting finite. Then, the quiver of A does not have a subquiver of Euclidean type $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ or $\widetilde{\mathbb{E}}_8$, since the path algebra of such a subquiver is minimal τ-tilting infinite as we mentioned in Section 2.3. Moreover, the local hereditariness makes A to be an incidence algebra.

On the other hand, it is shown in [Dra, Theorem 3.3] that an incidence algebra is strongly simply connected if and only if it does not contain a full subcategory whose quiver is a crown, i.e., is of the form



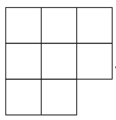
It is obvious that a crown is just a quiver of Euclidean type $\widetilde{\mathbb{A}}_n$ with zigzag orientation. If a crown appears in the quiver of A, then the corresponding idempotent truncation of A would surject to the path algebra of the crown, contradicting our assumption that A is τ-tilting finite. Therefore, A is a strongly simply connected incidence algebra and the statement follows from Theorem 4.2.3. $\square$

4.3 Algebras with rectangle or triangle quiver

In this section, we first recall the constructions of staircase algebras $\mathcal{A}(\lambda)$ introduced by Boos [Bo], which are parameterized by partitions λ. As we will show below, one can see that an algebra presented by a rectangle quiver with all possible commutativity relations, is actually a staircase algebra $\mathcal{A}(\lambda)$ with $\lambda = (m^n)$. Similarly, we introduce the shifted-staircase algebra $\mathcal{A}^s(\lambda^s)$ parameterized by a shifted partition λ^s as a generalization of algebras presented by triangle quivers.

4.3.1 Staircase algebras

We recall that a partition $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_\ell)$ of n is a non-increasing sequence of positive integers such that $\sum_{i\in\mathbb{N}} \lambda_i = n$. We may merge same entries of λ by potencies, for example, $(3,3,2,1,1) = (3^2,2,1^2)$. We can visualize λ by the Young diagram $Y(\lambda)$, that is, a box-diagram of which the i-th row contains λ_i boxes. For example,

$Y(3^2,2) =$  .

Starting with $(1,1)$ in the top-left corner, we assign each of the boxes in $Y(\lambda)$ a coordinate (i,j) by increasing i from top to bottom and j from left to right.

Definition 4.3.1 ([Bo, Definition 3.1]). Let λ be a partition. We define Q_λ and I_λ by

- the vertices of Q_λ are given by the boxes appearing in $Y(\lambda)$;
- the arrows of Q_λ are given by all $(i,j) \to (i,j+1)$ and $(i,j) \to (i+1,j)$, whenever all these vertices are defined.
- I_λ is a two-sided ideal generated by all possible commutativity relations for all squares appearing in Q_λ.

Then, the bound quiver algebra $\mathcal{A}(\lambda) := KQ_\lambda/I_\lambda$ is called a staircase algebra.

Following the above example, let $\lambda = (3^2, 2)$, the associated quiver Q_λ is given by

$$\begin{tikzcd}
(1,1) \arrow[r, "\alpha_{1,1}"] \arrow[d, "\beta_{1,1}"'] & (1,2) \arrow[r, "\alpha_{1,2}"] \arrow[d, "\beta_{1,2}"'] & (1,3) \arrow[d, "\beta_{1,3}"'] \\
(2,1) \arrow[r, "\alpha_{2,1}"] \arrow[d, "\beta_{2,1}"'] & (2,2) \arrow[r, "\alpha_{2,2}"] \arrow[d, "\beta_{2,2}"'] & (2,3) \\
(3,1) \arrow[r, "\alpha_{3,1}"] & (3,2) &
\end{tikzcd}
\simeq
\begin{tikzcd}
\circ \arrow[r] \arrow[d] & \circ \arrow[r] \arrow[d] & \circ \arrow[d] \\
\circ \arrow[r] \arrow[d] & \circ \arrow[r] \arrow[d] & \circ \\
\circ \arrow[r] & \circ &
\end{tikzcd},$$

and the corresponding staircase algebra $\mathcal{A}(\lambda)$ is defined by

$$\mathcal{A}(\lambda) := KQ_\lambda / < \alpha_{1,1}\beta_{1,2} - \beta_{1,1}\alpha_{2,1}, \alpha_{1,2}\beta_{1,3} - \beta_{1,2}\alpha_{2,2}, \alpha_{2,1}\beta_{2,2} - \beta_{2,1}\alpha_{3,1} >.$$

We denote by λ^T the transposed partition of a partition λ, which is given by the columns of the Young diagram $Y(\lambda)$ from left to right. Then, $\mathcal{A}(\lambda)$ is isomorphic to $\mathcal{A}(\lambda^T)$. Moreover, $\mathcal{A}(\lambda)$ is a basic, connected, triangular, finite-dimensional K-algebra.

Proposition 4.3.2 ([Bo, Proposition 3.7])**.** *Let λ be a partition. Then, the staircase algebra $\mathcal{A}(\lambda)$ is strongly simply connected.*

This implies that $\mathcal{A}(\lambda)$ is τ-tilting finite if and only if it is representation-finite. Since the author of [Bo] has given a complete classification of the representation type of $\mathcal{A}(\lambda)$, we can understand all τ-tilting finite staircase algebras by quiver and relations.

Theorem 4.3.3 ([Bo, Theorem 4.5])**.** *A staircase algebra $\mathcal{A}(\lambda)$ with a partition λ of n is*

(1) representation-finite ($\Leftrightarrow$ τ-tilting finite) if and only if one of the following holds:

- $\lambda \in \{(n), (n-k, 1^k), (n-2, 2), (2^2, 1^{n-4})\}$ *for* $k \leqslant n$.
- $n \leqslant 8$ *and* $\lambda \notin \{(4,3,1), (3^2,2), (3,2^2,1), (4,2,1^2)\}$.

(2) tame concealed if and only if λ comes up in the following list:

$$(6,3), (6,2,1), (5,2^2), (4,3,1), (4,2,1^2), (3,2^2,1), (3^2,1^3), (2^3,1^3), (3,2,1^4).$$

(3) tame, but not tame concealed if and only if λ comes up in the following list:

$$(5^2), (5,4), (4^2,1), (3^3), (3^2,2), (3,2^3), (2^5), (2^4,1).$$

Otherwise, $\mathcal{A}(\lambda)$ is wild.

Let $\vec{A}_n$ be the path algebra of Dynkin type $\mathbb{A}_n$ associated with linear orientation. Then, we define

$$\mathcal{B} := \left\{ B_{m,n} \mid B_{m,n} \text{ is the tensor product } \vec{A}_m \otimes_K \vec{A}_n \right\}.$$

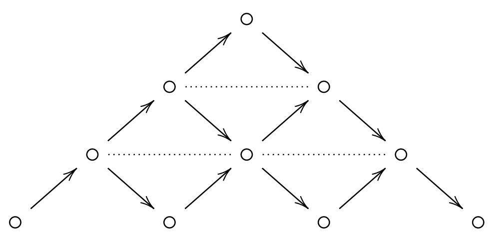
Note that $B_{m,n}$ is presented by a rectangle quiver with all possible commutativity relations and vice versa. In particular, $B_{m,n} \simeq B_{n,m}$. We also note that $B_{m,n}$ can be regarded as a special staircase algebra. Hence, we can determine the τ-tilting finiteness of $B_{m,n}$.

Corollary 4.3.4. *Let $B_{m,n} \in \mathcal{B}$. Then, the algebra $B_{m,n}$ is τ-tilting finite if and only if (m,n) or $(n,m) \in \{(1,k), (2,2), (2,3), (2,4) \mid k \in \mathbb{N}\}$.*

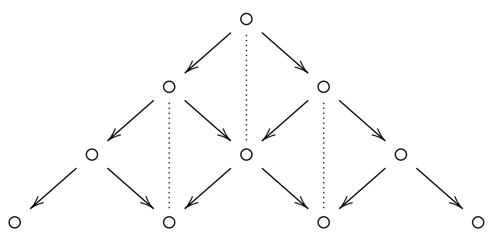
Proof. It is obvious that $B_{m,n}$ is a staircase algebra $\mathcal{A}(\lambda)$ with $\lambda = (m^n)$ or (n^m). Then, the statement follows from Theorem 4.3.3. $\square$

4.3.2 Shifted-staircase algebras

Now, we consider the triangle quivers. We point out that the quiver of staircase algebras cannot be a triangle quiver because of the different orientations. One may look at the following case as an example.

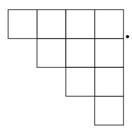


A triangle quiver Q_{tri}



$\mathcal{A}(\lambda)$ with $\lambda = (4, 3, 2, 1)$

This motivates us to introduce the shifted-staircase algebras. We recall that a shifted partition $\lambda^s = (\lambda_1^s > \lambda_2^s > \cdots > \lambda_\ell^s)$ is a strictly decreasing sequence of positive integers. We can visualize λ^s by the shifted Young diagram $Y(\lambda^s)$, that is, a box-diagram of which the i-th row contains λ_i boxes and is shifted to the right $i-1$ steps. For example, let $\lambda^s = (4, 3, 2, 1)$, then

$Y(\lambda^s) =$  .

Starting with $(1, 1)$ in the top-left corner, we assign each of the boxes in $Y(\lambda^s)$ a coordinate (i, j) with $j \geqslant i$ by increasing i from top to bottom and j from left to right.

Definition 4.3.5. Let λ^s be a shifted partition and $\mathcal{A}^s(\lambda^s) := KQ_{\lambda^s}/I_{\lambda^s}$ such that

- the vertices of Q_{λ^s} are given by the boxes appearing in $Y(\lambda^s)$;
- the arrows of Q_{λ^s} are given by all $(i, j) \to (i, j+1)$ and $(i, j) \to (i+1, j)$, whenever all these vertices are defined.
- I_{λ^s} is a two-sided ideal generated by all possible commutativity relations for all squares appearing in Q_{λ^s}.

Then, the bound quiver algebra $\mathcal{A}^s(\lambda^s)$ is called a shifted-staircase algebra.

For example, the quiver Q_{λ^s} with $\lambda^s = (4, 3, 2, 1)$ is exactly the triangle quiver Q_{tri} displayed above. In fact, the algebra presented by a triangle quiver with $\frac{n(n+1)}{2}$ vertices (and all possible commutativity relations), is exactly the shifted-staircase algebra $\mathcal{A}^s(\lambda^s)$ with $\lambda^s = (n, n-1, \ldots, 2, 1)$.

It is obvious that $\mathcal{A}^s(\lambda^s)$ is also a basic, connected, triangular, finite-dimensional K-algebra. Next, we show that $\mathcal{A}^s(\lambda^s)$ is strongly simply connected.

Proposition 4.3.6. *For a shifted partition λ^s, $\mathcal{A}^s(\lambda^s)$ is strongly simply connected.*

Proof. Let B be a convex subcategory of $\mathcal{A}^s(\lambda^s)$ and $P_{(i,j)}$ the indecomposable projective B-module at vertex (i,j). Then, one can check that $\mathsf{rad}\ P_{(i,j)}$ is either indecomposable or decomposed into exactly two indecomposable B-modules. The latter case appears if and only if $(i,j+1)$ and $(i+1,j)$ are vertices of the quiver Q_B of B, but $(i+1,j+1)$ is not.

For the former case, there is nothing to prove. For the latter case, let $Q(i,j)$ be the subquiver of Q_B obtained by deleting all vertices of Q_B being a source of a path in Q_B with target (i,j). Then, $Q(i,j)$ is decomposed into two disjoint subquivers such that $\mathsf{rad}\ P_{(i,j)}$ is separated. Hence, B satisfies the separation condition and it is simply connected following Proposition 4.1.7. Then, the statement follows from the definition of strongly simply connected algebras, see Definition 4.1.4. $\square$

Now, we have understood that $\mathcal{A}^s(\lambda^s)$ is τ-tilting finite if and only if it is representation-finite. In order to classify τ-tilting finite shifted-staircase algebras by quiver and relations, it is enough to give a complete classification for the representation type of shifted-staircase algebras. Before to do this, we need the following observation.

Let $\lambda^s = (\lambda_1^s > \lambda_2^s > \cdots > \lambda_\ell^s)$ and $\mu^s = (\mu_1^s > \mu_2^s > \cdots > \mu_k^s)$ be two shifted partitions. We say that $\lambda^s \leq \mu^s$ if $\ell \leqslant k$ and $\lambda_i^s \leqslant \mu_i^s$ for all $1 \leqslant i \leqslant \ell$.

Proposition 4.3.7. *Suppose $\lambda^s \leq \mu^s$. Then, $\mathcal{A}^s(\lambda^s)$ is a convex subcategory of $\mathcal{A}^s(\mu^s)$.*

We may use an example to understand the above proposition. Let $\lambda^s = (4,3,1)$ and $\mu^s = (4,3,2,1)$. Then, $\mathcal{A}^s(\lambda^s)$ is presented by

$$\begin{tikzcd}
\circ \arrow[r] & \circ \arrow[r] \arrow[d] \arrow[dr, dotted, no head] & \circ \arrow[r] \arrow[d] \arrow[dr, dotted, no head] & \circ \arrow[d] \\
 & \circ \arrow[r] & \circ \arrow[r] \arrow[d] & \circ \\
 & & \circ &
\end{tikzcd}$$

with all possible commutativity relations, while $\mathcal{A}^s(\mu^s)$ is presented by the triangle quiver Q_{tri} with all possible commutativity relations. One can easily find that $\mathcal{A}^s(\lambda^s)$ is a proper convex subcategory of $\mathcal{A}^s(\mu^s)$.

Theorem 4.3.8. *For a shifted partition λ^s, the shifted-staircase algebra $\mathcal{A}^s(\lambda^s)$ is*

- *representation-finite if and only if λ^s is one of (n), $(m-1,1)$ with $m \geqslant 3$, $(3,2)$, $(4,2)$, $(5,2)$, $(6,2)$, $(4,3)$, $(5,3)$, $(5,4)$, $(3,2,1)$ and $(4,2,1)$.*
- *tame concealed if and only if λ^s is one of $(6,3)$, $(7,2)$ and $(5,2,1)$.*
- *tame non-concealed if and only if λ^s is one of $(6,4)$, $(6,5)$, $(4,3,1)$, $(4,3,2)$ and $(4,3,2,1)$.*

Otherwise, $\mathcal{A}^s(\lambda^s)$ is wild.

Proof. We first observe that $\mathcal{A}^s(n)$ and $\mathcal{A}^s(m-1,1)$ with $m \geqslant 3$ are path algebras of Dynkin types $\mathbb{A}_n$ and $\mathbb{D}_m$, respectively. Thus, both of them are representation-finite.

Next, we directly construct the Tits form of $\mathcal{A}^s(\lambda^s)$ for some small cases. Then, we can use Proposition 4.1.9 to check their representation type since $\mathcal{A}^s(\lambda^s)$ is strongly simply connected. In particular, we use the software GAP to check whether a Tits form is weakly non-negative (resp., weakly positive) or not, where the method in GAP is introduced by [DP] (resp., [Ho]). Although we can also check that $\mathcal{A}^s(\lambda^s)$ is critical or not by checking the Tits form of all convex subcategories of $\mathcal{A}^s(\lambda^s)$, we would like to trust the list of critical algebras in [B4] since the list is also given independently in [HV].

(1) Assume that $Q_{(8,2)}$ is labeled as

$$\begin{tikzcd}
1 \arrow[r] & 2 \arrow[r] \arrow[d] \arrow[dr, dotted, no head] & 3 \arrow[r] \arrow[d] & 4 \arrow[r] & 5 \arrow[r] & 6 \arrow[r] & 7 \arrow[r] & 8 . \\
& 9 \arrow[r] & 10
\end{tikzcd}$$

By the definition of Tits form, we have

$$\begin{aligned} q_{\mathcal{A}^s(8,2)}(v) = v_1^2 - v_1v_2 + v_2^2 - v_2v_3 - v_2v_9 + v_2v_{10} + v_3^2 \\ - v_3v_4 - v_3v_{10} + v_4^2 - v_4v_5 + v_5^2 - v_5v_6 + v_6^2 \\ - v_6v_7 + v_7^2 - v_7v_8 + v_8^2 + v_9^2 - v_9v_{10} + v_{10}^2. \end{aligned}$$

Then, one can check that

$$q_{\mathcal{A}^s(8,2)} \begin{pmatrix} 10 & 20 & 25 & 20 & 16 & 12 & 5 & 1 \\ & 15 & 10 & & & & & \end{pmatrix} = -1.$$

We deduce that $\mathcal{A}^s(8,2)$ is wild following Proposition 4.1.9. Then, we observe that $\mathcal{A}^s(7,2)$ is the critical algebra numbered 18 in [B4] and hence, $\mathcal{A}^s(6,2)$, $\mathcal{A}^s(5,2)$, $\mathcal{A}^s(4,2)$ and $\mathcal{A}^s(3,2)$ are representation-finite. Similarly, one can check that $\mathcal{A}^s(7,3)$ is wild by

$$q_{\mathcal{A}^s(7,3)} \begin{pmatrix} 2 & 4 & 6 & 6 & 4 & 2 & 1 \\ & 4 & 4 & 2 & & & \end{pmatrix} = -1,$$

and $\mathcal{A}^s(6,3)$ is the critical algebra numbered 93 in [B4]. Then, $\mathcal{A}^s(5,3)$ and $\mathcal{A}^s(4,3)$ are representation-finite.

(2) Assume that $Q_{(6,5)}$ is labeled as

$$\begin{tikzcd}
1 \arrow[r] & 2 \arrow[r] \arrow[d] \arrow[dr, dotted, no head] & 3 \arrow[r] \arrow[d] \arrow[dr, dotted, no head] & 4 \arrow[r] \arrow[d] \arrow[dr, dotted, no head] & 5 \arrow[r] \arrow[d] \arrow[dr, dotted, no head] & 6 \arrow[d] . \\
& 7 \arrow[r] & 8 \arrow[r] & 9 \arrow[r] & 10 \arrow[r] & 11
\end{tikzcd}$$

Then, the Tits form $q_{\mathcal{A}^s(6,5)}(v) = vXv^T$ is given by

$$X = \frac{1}{2} \cdot \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

It can be checked by GAP that $q_{\mathcal{A}^s(6,5)}(v)$ is weakly non-negative, so that $\mathcal{A}^s(6,5)$ is not wild by Proposition 4.1.9. On the other hand, $\mathcal{A}^s(6,5)$ is representation-infinite since it has the following critical algebra as a proper convex subcategory,

$$\begin{tikzcd}
 & \circ \arrow[r] \arrow[d] \arrow[dr, dotted, no head] & \circ \arrow[r] \arrow[d] \arrow[dr, dotted, no head] & \circ \arrow[r] \arrow[d] & \circ \\
\circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ &
\end{tikzcd},$$

which is numbered 86 in [B4]. Thus, $\mathcal{A}^s(6,5)$ is tame non-concealed. We find that $\mathcal{A}^s(6,4)$ is also tame non-concealed since such a proper critical convex subcategory in $\mathcal{A}^s(6,5)$ remains in $\mathcal{A}^s(6,4)$. Then, we find that $\mathcal{A}^s(5,4)$ is representation-finite since the Tits form $q_{\mathcal{A}^s(5,4)}(v)$ is weakly positive.

(3) We point out that $\mathcal{A}^s(6,2,1)$ is wild by

$$q_{\mathcal{A}^s(6,2,1)}\begin{pmatrix} 2 & 4 & 6 & 4 & 2 & 1 \\ & 4 & 4 & & & \\ & & 2 & & & \end{pmatrix} = -1,$$

and $\mathcal{A}^s(5,2,1)$ is the critical algebra numbered 14 in [B4]. Therefore, $\mathcal{A}^s(4,2,1)$ and $\mathcal{A}^s(3,2,1)$ are representation-finite. Similarly, $\mathcal{A}^s(5,3,1)$ is wild by

$$q_{\mathcal{A}^s(5,3,1)}\begin{pmatrix} 1 & 1 & 3 & 4 & 2 \\ & 2 & 4 & 3 & \\ & & 2 & & \end{pmatrix} = -1.$$

Since $\mathcal{A}^s(4,3,2,1)$, $\mathcal{A}^s(4,3,2)$ and $\mathcal{A}^s(4,3,1)$ contain the following critical algebra (see [B4, Lemma 3.1]) as a proper convex subcategory,

$$\begin{tikzcd}
\circ \arrow[dr] & & \circ \arrow[dl] \arrow[dr] \arrow[dd, dotted, no head] & \\
 & \circ \arrow[dl] \arrow[dr] & & \circ \arrow[dl] \\
\circ & & \circ &
\end{tikzcd}$$

they are representation-infinite and not tame concealed. Similar to the case $\mathcal{A}^s(6,5)$, the Tits form $q_{\mathcal{A}^s(4,3,2,1)}(v)$ is weakly non-negative such that $\mathcal{A}^s(4,3,2,1)$, $\mathcal{A}^s(4,3,2)$ and $\mathcal{A}^s(4,3,1)$ are not wild.

Lastly, we observe that if λ^s does not contain one of $(8,2),(7,3),(6,5),(6,2,1)$ and $(5,3,1)$, then λ^s is listed in Theorem 4.3.8. Thus, all of the remaining cases are wild. $\square$

Remark 4.3.9. In fact, $\mathcal{A}^s(8,2)$ is a minimal wild[3] algebra which is the 3rd algebra in the first line on page 151 of [Un]; $\mathcal{A}^s(7,3)$ is the 4th minimal wild concealed algebra in the second line on page 152 of [Un]; $\mathcal{A}^s(6,2,1)$ is the 1st minimal wild concealed algebra in the second line on page 150 of [Un]; $\mathcal{A}^s(5,4)$ is a representation-finite sincere simply connected algebra, which is numbered 920 of type $(5,2,1)$ in [RT].

Corollary 4.3.10. *A shifted-staircase algebra $\mathcal{A}^s(\lambda^s)$ is τ-tilting finite if and only if the shifted partition λ^s is one of (n), $(m-1,1)$ with $m\geqslant 3$, $(3,2)$, $(4,2)$, $(5,2)$, $(6,2)$, $(4,3)$, $(5,3)$, $(5,4)$, $(3,2,1)$ and $(4,2,1)$.*

[3] A is called minimal wild if A is wild, but A/AeA is not wild for any non-zero idempotent e of A.

We define

$$\mathcal{C} := \left\{ C_n \mid \begin{array}{c} C_n \text{ is the algebra presented by a triangle quiver with} \\ \frac{n(n+1)}{2} \text{ vertices and all possible commutativity relations} \end{array} \right\}.$$

Corollary 4.3.11. *Let $C_n \in \mathcal{C}$. Then, C_n is τ-tilting finite if and only if $n \leqslant 3$.*

Proof. It is obvious that C_n is the algebra $\mathcal{A}^s(\lambda^s)$ with $\lambda^s := (n, n-1, \dots, 2, 1)$. □

At the end of this section, we may distinguish the following special cases. Let $\vec{A}_n$ be the path algebra of Dynkin type $\mathbb{A}_n$ associated with linear orientation. We define

$$\mathcal{D} := \left\{ D_n \mid D_n \text{ is the tensor product } \vec{A}_{2n+1} \otimes_K \vec{A}_n \right\}$$

and

$$\mathcal{E} := \left\{ E_n \mid E_n \text{ is the Auslander algebra of } \vec{A}_{2n} \right\}.$$

Then, D_n is the staircase algebra $\mathcal{A}(\lambda)$ with $\lambda = (n^{2n+1})$ and E_n is a quotient algebra of the shifted-staircase algebra $\mathcal{A}^s(\lambda^s)$ with $\lambda^s = (2n, 2n-1, \dots, 1)$, modulo some monomial relations. It is shown in [La, Corollary 1.13] that $D_n \in \mathcal{D}$ is derived equivalent to $E_n \in \mathcal{E}$.

Remark 4.3.12. Prof. Ariki pointed out that the derived equivalence between D_2 and E_2 gives an example that derived equivalence does not necessarily preserve the τ-tilting finiteness. Indeed, Proposition 4.2.11 implies that E_2 is τ-tilting finite because $T_2(\vec{A}_4) = \vec{A}_2 \otimes \vec{A}_4$ is τ-tilting finite by Corollary 4.3.4, while D_2 is τ-tilting infinite by Corollary 4.3.4.

Chapter 5

Schur Algebras

In this chapter, we focus on Schur algebras which play an important role in the theory of Schur-Weyl duality. In other words, this class of algebras links representations of the symmetric group G_r with representations of the general linear group $\mathsf{GL}_n(\mathbb{F})$ over a field $\mathbb{F}$. Since the class of Schur algebras was introduced, it has always received widespread attention and achieved significant influences in representation theory and Lie theory until now. For example, many derivatives appeared, such as q-Schur algebras, infinitesimal Schur algebras, Borel-Schur algebras and so on. In particular, the representation type of Schur algebras is completely determined by various authors, including Erdmann [Er], Xi [Xi], Doty-Nakano [DN] and Doty-Erdmann-Martin-Nakano [DEMN].

Let n, r be two positive integers and $\mathbb{F}$ an algebraically closed field of characteristic p. We take an n-dimensional vector space V over $\mathbb{F}$ with a basis $\{v_1, v_2, \ldots, v_n\}$. We denote by $V^{\otimes r}$ the r-fold tensor product $V \otimes_{\mathbb{F}} V \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} V$. Then, $V^{\otimes r}$ has a $\mathbb{F}$-basis given by

$$\{v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r} \mid 1 \leqslant i_j \leqslant n \text{ for all } 1 \leqslant j \leqslant r\}.$$

Let G_r be the symmetric group on r symbols and $\mathbb{F}G_r$ its group algebra. Then, G_r, and hence also $\mathbb{F}G_r$, act on the right on $V^{\otimes r}$ by place permutations of the subscripts, that is, for any $\sigma \in G_r$,

$$(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r}) \cdot \sigma = v_{i_{\sigma(1)}} \otimes v_{i_{\sigma(2)}} \otimes \cdots \otimes v_{i_{\sigma(r)}}.$$

We call the endomorphism ring $\mathsf{End}_{\mathbb{F}G_r}(V^{\otimes r})$ the Schur algebra (see [Ma, Section 2]) and denote it by $S_{\mathbb{F}}(n, r)$, or simply by $S(n, r)$.

In the first section, we recall some basic materials for the symmetric group G_r and the Schur algebra $S(n, r)$. Moreover, we give two reduction theorems such that we only need to consider small n and r, and we explain our strategy to prove $S(n, r)$ to be τ-tilting infinite. In the second section, we determine the number $\#\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ for a representation-finite block A, or a tame block A of a tame Schur algebra $S(n, r)$. As a consequence, we deduce that all tame Schur algebras are τ-tilting finite. In the last section, we determine the τ-tilting finiteness of wild Schur algebras.

5.1 Symmetric groups and Schur algebras

In this section, we review some of the backgrounds that will be needed in this chapter. Basically, we refer to some textbooks, such as [Ja], [Ma] and [Sa], for more details on the representation theory of the symmetric group and the Schur algebra.

Let r be a natural number and $\lambda := (\lambda_1, \lambda_2, \ldots)$ a sequence of non-negative integers. We call λ a partition of r if $\sum_{i \in \mathbb{N}} \lambda_i = r$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$, and the elements λ_i are called parts of λ. If there exists an $n \in \mathbb{N}$ such that $\lambda_i = 0$ for all $i > n$, then we denote λ by $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and call it a partition of r with at most n parts. We denote by $\Omega(r)$ the set of all partitions of r and by $\Omega(n, r)$ the set of all partitions of r with at most n parts. For example, $\Omega(5) = \{(5), (4,1), (3,2), (3,1^2), (2^2,1), (2,1^3), (1^5)\}$ and $\Omega(3,5) = \{(5), (4,1), (3,2), (3,1^2), (2^2,1)\}$.

Definition 5.1.1. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ be partitions of r. We say that λ dominates μ if

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \geqslant \mu_1 + \mu_2 + \cdots + \mu_i$$

for any $i \geqslant 1$, and we denote by $\lambda \trianglerighteq \mu$. We say that $\lambda > \mu$ in lexicographic order if there exists an index i satisfying $\lambda_i > \mu_i$ while $\lambda_j = \mu_j$ for any $j < i$.

As mentioned in the previous chapter, we may regard a partition λ of r as a box-diagram $[\lambda]$ of which the i-th row contains λ_i-boxes. For example,

$$[(5,3,2,1)] = \ydiagram{5,3,2,1}\ .$$

For a prime p, a partition λ or a diagram $[\lambda]$ is called p-regular if no p rows of λ have the same length. Otherwise, λ or $[\lambda]$ is called p-singular.

We can associate each box x of $[\lambda]$ with a hook H_x, which is the set of boxes below x, boxes on the right of x and x itself. Then, the hook length $|H_x|$ is defined as the number of boxes in H_x. In the above example, the hook lengths are

$$\begin{ytableau} 8 & 6 & 4 & 2 & 1 \\ 5 & 3 & 1 \\ 3 & 1 \\ 1 \end{ytableau}\ .$$

On the other hand, a hook H_x is called a p-hook if $|H_x| = p$. Then, we may get the p-core of $[\lambda]$ (or λ) by removing as many p-hooks as we can. For example, the 3-core of $\lambda = (5,3,2,1)$ is $\ydiagram{2}$, which is obtained by the following process

$$\begin{ytableau} 8 & 6 & 4 & 2 & 1 \\ 5 & 3 & 1 \\ 3 & 1 \\ 1 \end{ytableau} \longrightarrow \begin{ytableau} 6 & 5 & 4 & 2 & 1 \\ 3 & 2 & 1 \end{ytableau} \longrightarrow \begin{ytableau} 5 & 4 & 3 & 2 & 1 \end{ytableau} \longrightarrow \begin{ytableau} 2 & 1 \end{ytableau}.$$

It can be shown that the p-core of a partition λ is independent of the order in which p-hooks are removed.

5.1.1 Permutation modules

Let λ be a partition of r. A λ-tableau t is obtained from $[\lambda]$ by filling the boxes by numbers $\{1, 2, \ldots, r\}$ without repetition. In fact, t is a bijection between the boxes in $[\lambda]$ and the numbers in $\{1, 2, \ldots, r\}$. For any $\sigma \in G_r$, we define an action $t \cdot \sigma := t \circ \sigma$ by the composition of the bijection t and the permutation σ. Then, the column stabilizer of a λ-tableau t is defined as the subgroup C_t of G_r consisting of permutations preserving the numbers in each column of t. Similarly, the row stabilizer of t is the subgroup R_t consisting of permutations preserving the numbers in each row of t.

Let t, t' be two λ-tableaux. We may define a row-equivalence relation $t \sim t'$ if $t' = t \cdot \sigma$ for a $\sigma \in R_t$. The equivalence class of t under $\sim$ is called a λ-tabloid and is denoted by $\{t\}$. We also define a G_r-action on a λ-tabloid $\{t\}$ by $\{t\} \cdot \sigma := \{t \cdot \sigma\}$ for any $\sigma \in G_r$ and this action is well-defined. Then, the λ-polytabloid e_t associated with a λ-tableau t is defined by $e_t := \{t\} \cdot \kappa_t$, where $\kappa_t := \sum_{\sigma \in C_t} \mathrm{sgn}(\sigma)\sigma$ is the signed column sum.

To illustrate our construction above, we give the following example.

Example 5.1.2. Let $\lambda = (2, 1)$. Then, a complete list of λ-tableaux is

$$\left\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \right\}.$$

For $t = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$, we have $C_t = \{id, (13)\}$ and $R_t = \{id, (12)\}$. Then, the λ-tabloid $\{t\}$ is

$$\{t\} = \begin{array}{cc}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} = \left\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \right\},$$

and the λ-polytabloid e_t is

$$e_t = \{t\} - \{t\} \cdot (13) = \begin{array}{cc}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} - \begin{array}{cc}\hline 2 & 3 \\ \hline 1 & \\ \cline{1-1}\end{array}.$$

Let λ be a partition of r. We denote by M^λ the $\mathbb{F}$-vector space spanned by all λ-tabloids. Then, the G_r-action on λ-tabloids makes M^λ into a module over the group algebra $\mathbb{F}G_r$, which is cyclic and generated by any one λ-tabloid. We call M^λ the permutation module corresponding to λ. Moreover, it is clear from the definition that M^λ is the induced $\mathbb{F}G_r$-module $1_{G_\lambda} \uparrow^{G_r}$ for a Young subgroup G_λ of G_r, where $G_\lambda := G_{\{1,2,\ldots,\lambda_1\}} \times G_{\{\lambda_1+1,\lambda_1+2,\ldots,\lambda_1+\lambda_2\}} \times \cdots \times G_{\{\lambda_1+\cdots+\lambda_{n-1}+1,\lambda_1+\cdots+\lambda_{n-1}+2,\ldots,r\}}$, 1_{G_λ} denotes the trivial module for G_λ and $\uparrow$ denotes induction.

We define a bilinear form $\langle\ ,\ \rangle$ on the set of all λ-tabloids as follows,

$$\langle \{t_1\}, \{t_2\} \rangle := \begin{cases} 1 & \text{if } \{t_1\} = \{t_2\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, it can be shown that $\langle\ ,\ \rangle$ is a symmetric G_r-invariant bilinear form on M^λ. For any $\mathbb{F}G_r$-submodule N of M^λ, we define $N^\perp := \big\{ x \in M^\lambda \mid \langle x, y \rangle = 0 \text{ for all } y \in N \big\}$. It is obvious that $N^\perp$ is again a $\mathbb{F}G_r$-submodule of M^λ.

Let $S(n,r) = \mathsf{End}_{\mathbb{F}G_r}(V^{\otimes r})$ be the Schur algebra. We would like to find the basic algebra of $S(n,r)$ so that we have to find all indecomposable pairwise non-isomorphic direct summands of $V^{\otimes r}$. As the first step, we recall (e.g., see [Ve, Section 1.6]) that M^λ can be regarded as (not necessarily indecomposable) direct summands of $V^{\otimes r}$. Therefore, we have the following algebra isomorphism,

$$S(n,r) \simeq \mathsf{End}_{\mathbb{F}G_r}\left(\bigoplus_{\lambda\in\Omega(n,r)} n_\lambda M^\lambda\right),$$

where $1 \leqslant n_\lambda \in \mathbb{N}$ is the number of compositions of r with at most n parts which are rearrangement of λ.

5.1.2 Specht modules and Young modules

In order to find all indecomposable pairwise non-isomorphic direct summands of M^λ, we need Specht modules. Following the conventions in [Ja], we call the submodule S^λ of M^λ spanned by all λ-polytabloids the Specht module corresponding to λ.

Theorem 5.1.3 ([Ja, Theorem 4.12, Theorem 11.5])**.** *Let $\mathbb{F}G_r$ be the group algebra of the symmetric group G_r. If $\mathbb{F}$ is a field of characteristic zero, then $\{S^\lambda \mid \lambda \in \Omega(r)\}$ is a complete set of pairwise non-isomorphic simple $\mathbb{F}G_r$-modules. If $\mathbb{F}$ is a field of prime characteristic p, then each Specht module S^λ with λ being p-regular has a unique (up to isomorphism) simple top $D^\lambda := S^\lambda/(S^\lambda \cap (S^\lambda)^\perp)$ and $\{D^\lambda \mid \lambda \in \Omega(r), \lambda \text{ is } p\text{-regular}\}$ is a complete set of pairwise non-isomorphic simple $\mathbb{F}G_r$-modules.*

In the case of a p-singular partition μ, all of the composition factors of S^μ are D^λ such that λ is a p-regular partition with $\lambda \rhd \mu$.

Let $\mathbb{F}$ be a field of prime characteristic p. The decomposition number $[S^\lambda : D^\mu]$ provides how many times each simple module D^μ occurs as a composition factor of the Specht module S^λ. If we run all partitions of r, then we get the decomposition matrix of $\mathbb{F}G_r$. Usually, we place the p-regular partitions in the decreasing order with respect to the lexicographic order and above all p-singular partitions. Then, the decomposition matrix of $\mathbb{F}G_r$ has the following form, see [Ja, Corollary 12.3].

$$\begin{array}{rc}
 & \overbrace{\hphantom{* \quad * \quad * \quad \dots \quad 1}}^{D^\mu,\ \mu\ p\text{-regular}} \\
\begin{array}{r} S^\lambda, \lambda\ p\text{-regular}\left\{\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right. \\ \\ S^\lambda, \lambda\ p\text{-singular}\left\{\begin{array}{c} \\ \\ \end{array}\right. \end{array} &
\begin{pmatrix}
1 & & & & \\
* & 1 & & O & \\
* & * & 1 & & \\
\vdots & \vdots & \vdots & \ddots & \\
* & * & * & \dots & 1 \\
- & - & - & - & - \\
* & * & * & \dots & * \\
* & * & * & \dots & *
\end{pmatrix}
\end{array}$$

We recall that a permutation module M^λ over $\mathbb{F}$ is liftable by a p-modular system and therefore, it has an associated ordinary character $\mathsf{ch}\ M^\lambda$. Let χ^λ be the ordinary character corresponding to λ ($\xleftrightarrow{1:1}$ Specht module S^λ) over a field of characteristic zero. Then,

$$\mathsf{ch}\ M^\lambda = \chi^\lambda + \sum_{\mu \rhd \lambda} k_\mu \chi^\mu$$

with multiplicities k_μ which can be zero. We decompose M^λ into a sum of indecomposable direct summands $\oplus_{i=1}^n Y_i$ for $n \in \mathbb{N}$. Obviously, each summand Y_i is also liftable and has an associated ordinary character $\mathsf{ch}\ Y_i$.

Definition 5.1.4. The unique direct summand Y_i which the ordinary character χ^λ occurs in $\mathsf{ch}\ Y_i$, is called the Young module corresponding to λ and is denoted by Y^λ.

It is well-known that the Young module Y^λ has a Specht filtration which is given by $Y^\lambda = Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_k = 0$ for some $k \in \mathbb{N}$ with each Z_i/Z_{i+1} isomorphic to a Specht module S^μ with $\mu \trianglerighteq \lambda$. Moreover, the Young module Y^λ is self-dual, that is, $Y^\lambda \simeq D(Y^\lambda)$ with respect to $D = \mathsf{Hom}_{\mathbb{F}}(-, \mathbb{F})$. In fact, $D(Y^\lambda)$ becomes a right $\mathbb{F}G_r$-module via $(f \cdot \sigma)(x) := f(x\sigma^{-1})$ for $f \in D(Y^\lambda)$, $\sigma \in G_r$ and $x \in Y^\lambda$.

Theorem 5.1.5 ([Ma, Section 4.6]). *The set* $\{Y^\lambda \mid \lambda \in \Omega(n, r)\}$ *is a complete set of indecomposable pairwise non-isomorphic direct summands of* $\{M^\lambda \mid \lambda \in \Omega(n, r)\}$.

Now, we are able to explain how to construct the basic algebra of the Schur algebra $S(n, r)$. Let B be a block of the group algebra $\mathbb{F}G_r$ labeled by a p-core ω. It is well-known that a partition λ belongs to B if and only if λ has the same p-core ω. Then, we define

$$S_B := \mathsf{End}_{\mathbb{F}G_r}\left(\bigoplus_{\lambda \in B \cap \Omega(n,r)} Y^\lambda \right)$$

and the basic algebra of $S(n, r)$ is given by $\bigoplus S_B$, where the sum is taken over all blocks of $\mathbb{F}G_r$. Moreover, S_B is a direct sum of blocks of the basic algebra of $S(n, r)$. We remark that if we consider the Young modules Y^λ for partitions λ of r with at most n parts, the Specht modules S^μ in the Specht filtration of Y^λ and the composition factors D^μ which appear in Y^λ are also corresponding to the partitions with at most n parts. (The reason is that $[S^\lambda : D^\mu] \neq 0 \Rightarrow \mu \trianglerighteq \lambda$.)

In order to understand the explicit structure of Y^λ, we need to know the decomposition matrix $[S^\lambda : D^\mu]$ and the filtration multiplicities $[Y^\lambda : S^\mu]$. Note that the latter one $[Y^\lambda : S^\mu]$ is equivalent to the ordinary character $\mathsf{ch}\ Y^\lambda$ of Y^λ. Then, it is worth mentioning that Henke [He] provided a formula to calculate $\mathsf{ch}\ Y^\lambda$ when λ is a partition with at most two parts. We recall these constructions as follows. Let p be a prime. There is a p-adic decomposition $s = \sum_{s \geqslant 0} s_k p^k$ for any non-negative integer s. Now, let s, t be two non-negative integers, we define a function

$$f(s, t) = \prod_{k \in \{0\} \cup \mathbb{N}} \binom{p - 1 - s_k}{p - 1 - t_k},$$

where we set $\binom{m}{n} = 0$ if $m < n$. Moreover, we have

$$g(s,t) := \begin{cases} 1 & \text{if } f(2t, s+t) = 1, \\ 0 & \text{otherwise.} \end{cases} \quad \text{and } h(s,t) := \begin{cases} 1 & \text{if } f(2t+1, s+t+1) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 5.1.6 ([He, Section 5.2])**.** *Let $(r-k,k)$ be a partition with a non-negative integer k and $\mathsf{ch}\ Y^{(r-k,k)}$ the associated ordinary character of $Y^{(r-k,k)}$.*

(1) If r is even, then

$$\mathsf{ch}\ Y^{(r-k,k)} = \sum_{i=0}^{\frac{r}{2}} g(\tfrac{r}{2} - i, \tfrac{r}{2} - k)\chi^{(r-i,i)}.$$

(2) If r is odd, then

$$\mathsf{ch}\ Y^{(r-k,k)} = \sum_{i=0}^{[\frac{r}{2}]} h([\tfrac{r}{2}] - i, [\tfrac{r}{2}] - k)\chi^{(r-i,i)},$$

where $[\frac{r}{2}]$ is the greatest integer less than or equal to $\frac{r}{2}$.

We may give an example to illustrate our constructions above.

Example 5.1.7. We look at the Schur algebra $S(2,11)$ over $p = 2$. Let B_1 be the principal block of $\mathbb{F}G_{11}$ and B_2 the block of $\mathbb{F}G_{11}$ labeled by 2-core $(2,1)$. Then, we may find in [Ja] that the parts of the decomposition matrix $[S^\lambda : D^\mu]$ for the partitions in B_1 and B_2 with at most two parts are

$$B_1 : \begin{array}{c} (11) \\ (9,2) \\ (7,4) \end{array} \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 1 & 0 & 1 \end{pmatrix}, \quad B_2 : \begin{array}{c} (10,1) \\ (8,3) \\ (6,5) \end{array} \begin{pmatrix} 1 & & \\ 1 & 1 & \\ 0 & 1 & 1 \end{pmatrix}.$$

We determine S_{B_2} as follows. By using the formula given in Lemma 5.1.6, we have

$$\begin{aligned} \mathsf{ch}\ Y^{(10,1)} &= \chi^{(10,1)}, \\ \mathsf{ch}\ Y^{(8,3)} &= \chi^{(10,1)} + \chi^{(8,3)}, \\ \mathsf{ch}\ Y^{(6,5)} &= \chi^{(10,1)} + \chi^{(8,3)} + \chi^{(6,5)}. \end{aligned}$$

Similar to the proof of [Er, Lemma 4.4], we may read off the Specht filtration of Y^λ from the formula, and the composition factors of Young modules are $\{D^{(10,1)}, D^{(8,3)}, D^{(6,5)}\}$. It is obvious that $Y^{(10,1)} = S^{(10,1)} = D^{(10,1)}$. By the decomposition matrix above, the Specht module $S^{(8,3)}$ has composition factors $\{D^{(10,1)}, D^{(8,3)}\}$. Since the top of $S^{(8,3)}$ is $D^{(8,3)}$ and $S^{(8,3)}$ is a submodule of $Y^{(8,3)}$, the simple module $D^{(10,1)}$ is in the socle of $Y^{(8,3)}$. We deduce the radical series of $Y^{(8,3)}$ by using the self-duality of Young modules, that is,

$$Y^{(8,3)} = \begin{array}{c} S^{(10,1)} \\ S^{(8,3)} \end{array} = \begin{array}{c} D^{(10,1)} \\ D^{(8,3)} \\ D^{(10,1)} \end{array}.$$

Similarly, the simple module $D^{(10,1)}$ appears in the top of $Y^{(6,5)}$ because $Y^{(6,5)}$ has Specht filtration whose top is $S^{(10,1)}$. As $(6,5)$ is 2-regular, the top of $S^{(6,5)}$ is $D^{(6,5)}$ and the socle of $S^{(6,5)}$ is $D^{(8,3)}$. Since $S^{(6,5)}$ is the bottom Specht module, it implies that $D^{(8,3)}$ appears in the socle of $Y^{(6,5)}$. By the self-duality of Young modules, we deduce that

$$Y^{(6,5)} = \begin{array}{ccccccc} & & D^{(8,3)} & & & & D^{(10,1)} \\ & / & & \diagdown & & / & \\ & & & & D^{(6,5)} & & \\ & & & & & \diagdown & \\ D^{(10,1)} & & & & & & D^{(8,3)} \end{array}.$$

Thus, $S_{B_2} = \mathsf{End}_{\mathbb{F}G_{11}}(Y^{(10,1)} \oplus Y^{(8,3)} \oplus Y^{(6,5)})$ is isomorphic to $\mathbb{F}Q/I$ with

$$Q: \ (10,1) \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} (6,5) \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} (8,3) \ \text{ and } I: \left\langle \alpha_1\beta_1, \beta_2\alpha_2, \alpha_1\alpha_2\beta_2, \alpha_2\beta_2\beta_1 \right\rangle,$$

where we replace each vertex in Q by λ associated with the Young module Y^λ.

Similarly, we use the formula given in Lemma 5.1.6 to calculate the characters of Young modules as follows,

$$\mathsf{ch}\ Y^{(11)} = \chi^{(11)},\ \mathsf{ch}\ Y^{(9,2)} = \chi^{(9,2)},\ \mathsf{ch}\ Y^{(7,4)} = \chi^{(11)} + \chi^{(7,4)}.$$

Then, we have

$$Y^{(11)} = D^{(11)}, Y^{(9,2)} = D^{(9,2)}, Y^{(7,4)} = \begin{array}{c} D^{(11)} \\ D^{(7,4)} \\ D^{(11)} \end{array},$$

and one may easily check that S_{B_1} is isomorphic to $\mathbb{F}Q/I \oplus \mathbb{F}$, where

$$Q: \ (11) \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} (7,4) \ \text{ and } I: \left\langle \alpha_1\beta_1 \right\rangle.$$

Therefore, the basic algebra of $S(2,11)$ over $p=2$ is $S_{B_1} \oplus S_{B_2}$.

5.1.3 Reduction theorems on Schur algebras

We give two useful reduction theorems which will allow us to simplify the general problem to the cases with small n and r. First of all, it is obvious from the previous constructions that $S(n,r)$ with $n > r$ is always Morita equivalent to $S(r,r)$. Then,

Lemma 5.1.8. *If $S(n,r)$ is τ-tilting infinite, then so is $S(N,r)$, for any $N > n$.*

Proof. Let S be the basic algebra of $S(N,r)$. For each $\lambda \in \Omega(n,r)$, we define e_λ to be the projector to Y^λ and take the sum $e := \sum e_\lambda$ over all partitions in $\Omega(n,r)$. Then, the idempotent truncation is

$$eSe = e\mathsf{End}_{\mathbb{F}G_r}\left(\bigoplus_{\lambda\in\Omega(N,r)} Y^\lambda\right)e = \mathsf{End}_{\mathbb{F}G_r}\left(\bigoplus_{\lambda\in\Omega(n,r)} Y^\lambda\right).$$

This implies that the basic algebra of $S(n,r)$ is an idempotent truncation of $S(N,r)$ for any $N > n$. Thus, the statement follows from Proposition 2.3.5. $\square$

We recall that the coordinate function $c_{ij} : \mathsf{GL}_n(\mathbb{F}) \to \mathbb{F}$ is defined by $c_{ij}(g) = g_{ij}$ for all $g = [g_{ij}] \in \mathsf{GL}_n(\mathbb{F})$, where $i, j \in \{1, 2, \dots, n\}$. Then, we denote by $A(n, r)$ the coalgebra generated by the homogeneous polynomials of total degree r in c_{ij}. In fact, the Schur algebra $S(n, r)$ is just the dual of $A(n, r)$.

Lemma 5.1.9. *If $S(n, r)$ is τ-tilting infinite, then so is $S(n, n + r)$.*

Proof. It has been proved in [Er] that $S(n, r)$ is a quotient of $S(n, n + r)$. We recall the proof as follows. Let $I = I(n, n + r)$ be the set of maps

$$\alpha : \{1, 2, \dots, n + r\} \to \{1, 2, \dots, n\}$$

with right G_{n+r}-action. Then, $S(n, n + r)$ has a basis $\{\xi_{\alpha,\beta} \mid (\alpha, \beta) \in (I \times I)/G_{n+r}\}$: the dual basis of $c_{\alpha(1)\beta(1)}c_{\alpha(2)\beta(2)} \cdots c_{\alpha(n+r)\beta(n+r)} \in A(n, n + r)$. Then, the elements $\xi_\alpha := \xi_{\alpha,\alpha}$ form a set of orthogonal idempotents for $S(n, n + r)$ whose sum is the identity. Note that $\Omega(n, n + r) \subseteq I$ is the set of representatives of G_{n+r}-orbits. Let $e = \sum \xi_\lambda$ be the idempotent of $S(n, n+r)$, where the sum is taken over all $\lambda \in \Omega(n, n+r)$ such that $\lambda_n = 0$. Then, by using $\mathsf{det}\ (c_{ij})$, we may get

$$S(n, n + r)/S(n, n + r)eS(n, n + r) \simeq S(n, r).$$

Therefore, the statement follows from Proposition 2.3.5. $\square$

5.1.4 Strategy on τ-tilting infinite Schur algebras

Let $A \simeq \mathbb{F}Q/I$ be an algebra presented by a quiver Q and an admissible ideal I. We call Q a *τ-tilting infinite quiver* if $A/\mathsf{rad}^2 A$ is τ-tilting infinite. For example, the Kronecker quiver $Q : \circ \rightrightarrows \circ$ is a τ-tilting infinite quiver, see Lemma 3.1.1. Then, the following lemma provides us with three τ-tilting infinite quivers.

Lemma 5.1.10. *The following quivers $\mathsf{Q}_1, \mathsf{Q}_2$ and Q_3 are τ-tilting infinite quivers,*

$$\mathsf{Q}_1 : \begin{tikzcd} \circ \arrow[r, shift left] \arrow[d, shift left] & \circ \arrow[l, shift left] \arrow[d, shift left] \\ \circ \arrow[r, shift left] \arrow[u, shift left] & \circ \arrow[l, shift left] \arrow[u, shift left] \end{tikzcd}, \quad \mathsf{Q}_2 : \begin{tikzcd} \circ \arrow[dr, shift left] & & \circ \arrow[dl, shift left] \\ \circ \arrow[r, shift left] & \circ \arrow[l, shift left] \arrow[r, shift left] \arrow[ul, shift left] \arrow[ur, shift left] & \circ \arrow[l, shift left] \end{tikzcd}, \quad \mathsf{Q}_3 : \begin{tikzcd} \circ \arrow[r, shift left] & \circ \arrow[l, shift left] \arrow[r, shift left] \arrow[d, shift left] & \circ \arrow[l, shift left] \\ \circ \arrow[r, shift left] & \circ \arrow[l, shift left] \arrow[r, shift left] \arrow[u, shift left] & \circ \arrow[l, shift left] \end{tikzcd}.$$

Proof. We look at the following subquivers,

$$\mathsf{Q}_1' : \begin{tikzcd} \circ \arrow[r] \arrow[d] & \circ \\ \circ & \circ \arrow[l] \arrow[u] \end{tikzcd}, \quad \mathsf{Q}_2' : \begin{tikzcd} \circ \arrow[dr] & & \circ \arrow[dl] \\ \circ \arrow[r] & \circ & \circ \arrow[l] \end{tikzcd}, \quad \mathsf{Q}_3' : \begin{tikzcd} \circ \arrow[r] & \circ & \circ \arrow[l] \\ \circ & \circ \arrow[l] \arrow[r] \arrow[u] & \circ \end{tikzcd}.$$

Since the path algebra of Q_i' for $i = 1, 2, 3$ is a quotient algebra of $A = \mathbb{F}Q/I$ if $Q = \mathsf{Q}_i$, and all of these path algebras are τ-tilting infinite as mentioned in Section 4.2, we conclude that $A/\mathsf{rad}^2 A$ is τ-tilting infinite if $Q = \mathsf{Q}_1, \mathsf{Q}_2, \mathsf{Q}_3$ by Proposition 2.3.5. $\square$

We remark that Adachi [Ad1] (and Aoki [Ao]) provided a handy criteria for the τ-tilting finiteness of radical square zero algebras, that is, for any algebra A, the quotient A/rad^2A is τ-tilting finite if and only if every single subquiver of the separated quiver for A/rad^2A is a disjoint union of Dynkin quivers. This also gives a proof of Lemma 5.1.10.

We mention that in order to show that $S(n,r)$ is τ-tilting infinite, it suffices to find a block algebra of $S(n,r)$ which is Morita equivalent to $\mathbb{F}Q/I$ with a τ-tilting infinite subquiver in Q. Then, the advantage is that it is not necessary to find the explicit relations in I. To find a τ-tilting infinite subquiver in $S(2,r)$, it is worth mentioning Erdmann and Henke's method [EH1]. Let $\lambda=(\lambda_1,\lambda_2)$ and $\mu=(\mu_1,\mu_2)$ be two partitions of r, we define two non-negative integers $s:=\lambda_1-\lambda_2$ and $t:=\mu_1-\mu_2$. We denote by v^s the vertex in the quiver of $S(2,r)$ corresponding to the Young module $Y^{(\lambda_1,\lambda_2)}$ with $s=\lambda_1-\lambda_2$. Let $n(v^s,v^t)$ be the number of arrows from v^s to v^t, then it is shown in [EH1, Theorem 3.1] that $n(v^s,v^t)=n(v^t,v^s)$ and $n(v^s,v^t)$ is either 0 or 1. We have the following recursive algorithm for computing $n(v^s,v^t)$.

Lemma 5.1.11 ([EH1, Proposition 3.1]). *Suppose that p is a prime characteristic and $s>t$. Let $s=s_0+ps'$ and $t=t_0+pt'$ with $0\leqslant s_0,t_0\leqslant p-1$ and $s',t'\geqslant 0$.*

(1) If $p=2$, then

$$n(v^s,v^t)=\begin{cases} n(v^{s'},v^{t'}) & \textit{if } s_0=t_0=1 \textit{ or } s_0=t_0=0 \textit{ and } s'\equiv t' \bmod 2,\\ 1 & \textit{if } s_0=t_0=0, t'+1=s'\not\equiv 0 \bmod 2,\\ 0 & \textit{otherwise.}\end{cases}$$

(2) If $p>2$, then

$$n(v^s,v^t)=\begin{cases} n(v^{s'},v^{t'}) & \textit{if } s_0=t_0,\\ 1 & \textit{if } s_0+t_0=p-2, t'+1=s'\not\equiv 0 \bmod p,\\ 0 & \textit{otherwise.}\end{cases}$$

5.2 Representation-finite and tame Schur algebras

In this section, we show that all tame Schur algebras are τ-tilting finite. We first recall the complete classification of the representation type of Schur algebras. Note that some semi-simple cases are contained in the representation-finite cases. We may distinguish the semi-simple cases following [DN]. Namely, the Schur algebra $S(n,r)$ is semi-simple if and only if $p=0$, or $p>r$, or $p=2,n=2,r=3$.

Proposition 5.2.1 ([Er], [DEMN]). *Let $p>0$ be the characteristic of $\mathbb{F}$. Then, the Schur algebra $S(n,r)$ is representation-finite if and only if $p=2,n=2,r=5,7$ or $p\geqslant 2,n=2,r<p^2$ or $p\geqslant 2,n\geqslant 3,r<2p$; tame if and only if $p=2,n=2,r=4,9,11$ or $p=3,n=2,r=9,10,11$ or $p=3,n=3,r=7,8$. Otherwise, $S(n,r)$ is wild.*

5.2.1 Representation-finite blocks

Erdmann [Er, Proposition 4.1] showed that each block A of a representation-finite Schur algebra $S(n,r)$ is Morita equivalent to $\mathcal{A}_m := \mathbb{F}Q/I$ for some $m \in \mathbb{N}$, which is defined by the following quiver and relations,

$$Q:\ 1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \cdots \underset{\beta_{m-2}}{\overset{\alpha_{m-2}}{\rightleftarrows}} m-1 \underset{\beta_{m-1}}{\overset{\alpha_{m-1}}{\rightleftarrows}} m\ ,$$
$$I : \langle \alpha_1\beta_1, \alpha_i\alpha_{i+1}, \beta_{i+1}\beta_i, \beta_i\alpha_i - \alpha_{i+1}\beta_{i+1} \mid 1 \leqslant i \leqslant m-2 \rangle.$$

Three years later after [Er], Donkin and Reiten [DR, Theorem 2.1] generalized this result to an arbitrary Schur algebra, that is, each representation-finite block of Schur algebras is Morita equivalent to $\mathcal{A}_m$ for some $m \in \mathbb{N}$.

We would like to determine the number of pairwise non-isomorphic basic support τ-tilting modules for a representation-finite block of Schur algebras.

Theorem 5.2.2. *Let $\mathcal{A}_m$ be the algebra defined above. Then, $\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ \mathcal{A}_m = \binom{2m}{m}$.*

Proof. Let Λ_m be the Brauer tree algebra whose Brauer tree is a straight line having $m+1$ vertices and without exceptional vertex. Then, it is easy to check that $\mathcal{A}_m$ is a quotient algebra of Λ_m modulo the two-sided ideal generated by $\alpha_1\beta_1$. Since $\alpha_1\beta_1$ is a central element of Λ_m and $\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ \Lambda_m = \binom{2m}{m}$ has been determined in [Ao, Theorem 5.6], we get the statement following Proposition 2.3.6. $\square$

5.2.2 Tame Schur algebras

The block algebras of tame Schur algebras are well-studied in [Er] and [DEMN]. In this subsection, we recall these constructions and show that tame Schur algebras are τ-tilting finite. We recall the following bound quiver algebras constructed in [Er], where the tameness for them is given in [DEMN, 5.5, 5.6, 5.7].

- Let $\mathcal{D}_3 := \mathbb{F}Q/I$ be the special biserial algebra given by

$$Q:\ \circ \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \circ \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \circ \ \text{ and } I : \Big\langle \alpha_1\beta_1, \beta_2\alpha_2, \alpha_1\alpha_2\beta_2, \alpha_2\beta_2\beta_1 \Big\rangle.$$

- Let $\mathcal{D}_4 := \mathbb{F}Q/I$ be the bound quiver algebra given by

$$Q:\ \begin{tikzcd} \circ \arrow[r, shift left, "\alpha_1"] & \circ \arrow[l, shift left, "\beta_1"] \arrow[r, shift left, "\beta_3"] \arrow[d, shift left, "\beta_2"] & \circ \arrow[l, shift left, "\alpha_3"] \\ & \circ \arrow[u, shift left, "\alpha_2"] & \end{tikzcd} \ \text{ and } I : \left\langle \begin{matrix} \alpha_1\beta_1, \alpha_2\beta_2, \alpha_3\beta_1, \alpha_3\beta_2, \alpha_1\beta_3, \alpha_2\beta_3, \\ \alpha_1\beta_2\alpha_2, \beta_2\alpha_2\beta_1, \beta_2\alpha_2 - \beta_3\alpha_3 \end{matrix} \right\rangle.$$

- Let $\mathcal{R}_4 := \mathbb{F}Q/I$ be the bound quiver algebra given by

$$Q:\ \circ \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \circ \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \circ \underset{\beta_3}{\overset{\alpha_3}{\rightleftarrows}} \circ \ \text{ and } I : \left\langle \begin{matrix} \alpha_1\beta_1, \alpha_1\alpha_2, \beta_2\beta_1, \\ \alpha_2\beta_2 - \beta_1\alpha_1, \alpha_3\beta_3 - \beta_2\alpha_2 \end{matrix} \right\rangle.$$

- Let $\mathcal{H}_4 := \mathbb{F}Q/I$ be the bound quiver algebra given by

$$Q:\ \begin{tikzcd} 1 \arrow[r, shift left, "\alpha_1"] & 2 \arrow[l, shift left, "\beta_1"] \arrow[r, shift left, "\alpha_3"] \arrow[d, shift left, "\beta_2"] & 4 \arrow[l, shift left, "\beta_3"] \\ & 3 \arrow[u, shift left, "\alpha_2"] & \end{tikzcd} \text{ and } I: \left\langle \begin{array}{c} \alpha_1\beta_1, \alpha_1\beta_2, \alpha_2\beta_1, \alpha_2\beta_2, \alpha_1\alpha_3, \\ \beta_3\beta_1, \alpha_3\beta_3 - \beta_1\alpha_1 - \beta_2\alpha_2 \end{array} \right\rangle.$$

We remark that $\mathcal{D}_3$, $\mathcal{D}_4$, $\mathcal{R}_4$ and $\mathcal{H}_4$ are also tame blocks of some wild Schur algebras.

Lemma 5.2.3. *The tame algebras $\mathcal{D}_3$, $\mathcal{D}_4$, $\mathcal{R}_4$ and $\mathcal{H}_4$ are τ-tilting finite. Moreover,*

A	$\mathcal{D}_3$	$\mathcal{D}_4$	$\mathcal{R}_4$	$\mathcal{H}_4$
$\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ A$	*28*	*114*	*88*	*96*

.

Proof. We often use Proposition 2.3.6 to reduce the direct calculation of left mutations.

(1) Since $\alpha_2\beta_2$ and $\beta_2\beta_1\alpha_1\alpha_2$ are non-trivial central elements of $\mathcal{D}_3$, we may define

$$\widetilde{\mathcal{D}_3} := \mathcal{D}_3/ < \alpha_2\beta_2, \beta_2\beta_1\alpha_1\alpha_2 >$$

so that $\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ \widetilde{\mathcal{D}_3} = \#\mathsf{s}\tau\text{-}\mathsf{tilt}\ \mathcal{D}_3$. Then, we determine the number $\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ \widetilde{\mathcal{D}_3}$ by calculating the left mutations starting with $\widetilde{\mathcal{D}_3}$. In fact, this is equivalent to finding the Hasse quiver $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ \widetilde{\mathcal{D}_3})$. We recall that the indecomposable projective $\widetilde{\mathcal{D}_3}$-modules are

$$P_1 = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}, P_2 = \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ 2 & & \\ 3 & & \end{smallmatrix}, P_3 = \begin{smallmatrix} 3 \\ 2 \\ 1 \\ 2 \end{smallmatrix}.$$

Starting with the unique maximal τ-tilting module $\widetilde{\mathcal{D}_3} \simeq P_1 \oplus P_2 \oplus P_3$, we take an exact sequence with a minimal left $\mathsf{add}(P_2 \oplus P_3)$-approximation f_1 of P_1:

$$\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \xrightarrow{f_1} \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ 2 & & \\ 3 & & \end{smallmatrix} \longrightarrow \mathsf{coker}\ f_1 \longrightarrow 0.$$

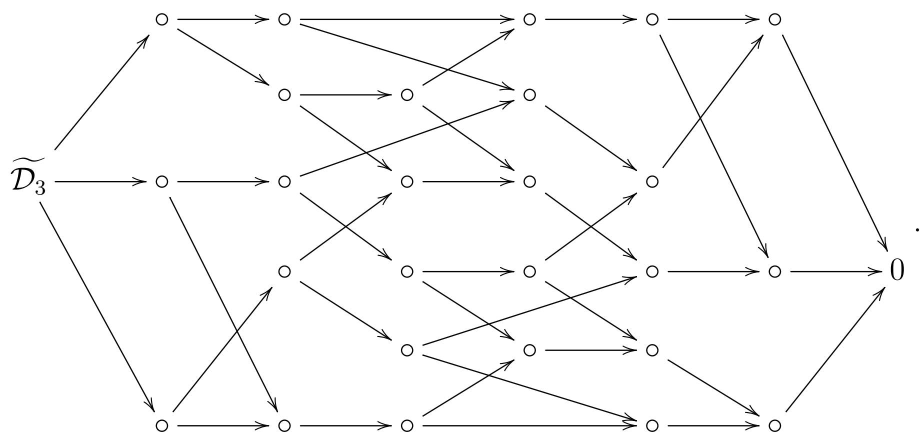
Then, $\mathsf{coker}\ f_1 = \begin{smallmatrix} 2 \\ 3 \end{smallmatrix}$ and $\mu_{P_1}^-(\widetilde{\mathcal{D}_3}) = \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus P_2 \oplus P_3$. Next, we take an exact sequence with a minimal left $\mathsf{add}(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus P_3)$-approximation f_2 of P_2:

$$P_2 \xrightarrow{f_2} \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus P_3 \longrightarrow \mathsf{coker}\ f_2 \longrightarrow 0,$$

so that $\mathsf{coker}\ f_2 = \begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$ and $\mu_{P_2}^-(\mu_{P_1}^-(\widetilde{\mathcal{D}_3})) = \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \oplus P_3$. Similarly, $\mu_{P_3}^-(\mu_{P_2}^-(\mu_{P_1}^-(\widetilde{\mathcal{D}_3}))) = \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$. Then, by the calculation in Example 2.1.7, we have

$$\begin{tikzcd} & 3 \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \arrow[r] & 3 \arrow[dr] & \\ \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \arrow[ur] \arrow[dr] & & & 0 \\ & \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus 2 \arrow[r] & 2 \arrow[ur] & \end{tikzcd}.$$

In this way, one can calculate all possible left mutation sequences starting with $\widetilde{\mathcal{D}_3}$ and ending at 0, so that the Hasse quiver $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ \widetilde{\mathcal{D}_3})$ is as follows,



Hence, we deduce that $\mathcal{D}_3$ is τ-tilting finite and $\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ \mathcal{D}_3 = 28$.

(2) Since $\beta_2\alpha_2$, $\alpha_3\beta_3$ and $\alpha_2\beta_1\alpha_1\beta_2$ are central elements of $\mathcal{D}_4$, we define

$$\widetilde{\mathcal{D}_4} := \mathcal{D}_4 / < \beta_2\alpha_2, \alpha_3\beta_3, \alpha_2\beta_1\alpha_1\beta_2 >.$$

Then, $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ \mathcal{D}_4) \simeq \mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ \widetilde{\mathcal{D}_4})$ by Proposition 2.3.6. Note that $\widetilde{\mathcal{D}_4}$ is just the algebra we dealt with in Example 2.3.10, we have $\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ \mathcal{D}_4 = 114$.

(3) Since $\beta_1\alpha_1$ and $\beta_2\alpha_2 + \beta_3\alpha_3$ are non-trivial central elements of $\mathcal{R}_4$, we define

$$\widetilde{\mathcal{R}_4} := \mathcal{R}_4 / < \beta_1\alpha_1, \beta_2\alpha_2 + \beta_3\alpha_3 >$$

and then, $\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ \widetilde{\mathcal{R}_4} = \#\mathsf{s}\tau\text{-}\mathsf{tilt}\ \mathcal{R}_4$. Instead of direct calculation, we point out that $\widetilde{\mathcal{R}_4}$ is a representation-finite string algebra and $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ \widetilde{\mathcal{R}_4})$ can be constructed by the String Applet [Geu]. Thus, we deduce that $\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ \mathcal{R}_4 = 88$.

(4) We recall that $\beta_1\alpha_1$ and $\beta_2\alpha_2 + \beta_3\alpha_3$ are central elements of $\mathcal{H}_4$. Then, we define

$$\widetilde{\mathcal{H}_4} := \mathcal{H}_4 / < \beta_1\alpha_1, \beta_2\alpha_2 + \beta_3\alpha_3 >.$$

Similar to the strategy in Example 2.3.10, we determine the number $\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ \widetilde{\mathcal{H}_4}$ step by step. First, we have $a_0(\widetilde{\mathcal{H}_4}) = 1$ and $a_1(\widetilde{\mathcal{H}_4}) = 4$.

Let M be a support τ-tilting $\widetilde{\mathcal{H}_4}$-module with support-rank 2, and with supports e_i and e_j $(i \neq j)$. Then, M becomes a τ-tilting $\widetilde{\mathcal{H}_4}/J$-module with $J :=< 1 - e_i - e_j >$. We denote by $\mathsf{b}_{i,j}$ the number of τ-tilting $\widetilde{\mathcal{H}_4}/J$-modules. Then, it is easy to check that

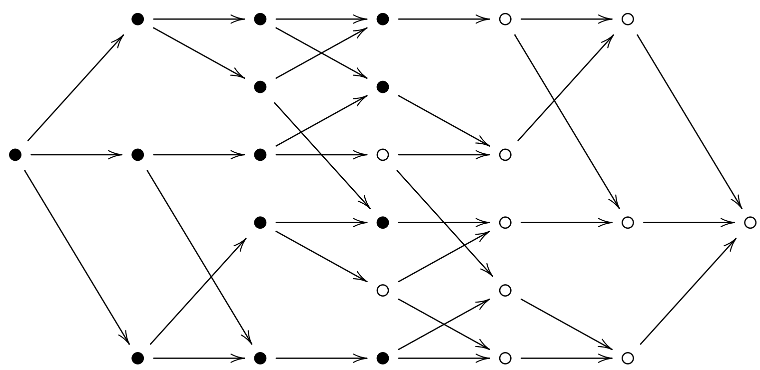
(i,j)	$(1,2)$	$(1,3)$	$(1,4)$	$(2,3)$	$(2,4)$	$(3,4)$
$\mathsf{b}_{i,j}$	3	1	1	3	3	1

This implies that $a_2(\widetilde{\mathcal{H}_4}) = 12$.

Let N be a support τ-tilting $\widetilde{\mathcal{H}_4}$-module with support-rank 3. Then, N becomes a τ-tilting $\widetilde{\mathcal{H}_4}/L_j$-module with $L_j :=< e_j >$, where e_j is the only one non-zero primitive idempotent satisfying $Ne_j = 0$. We denote by d_j the number of τ-tilting $\widetilde{\mathcal{H}_4}/L_j$-modules. For example, if $j = 1$, then

$$\widetilde{\mathcal{H}_4}/L_1 := \mathbb{F}\left(3 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} 2 \underset{\beta_3}{\overset{\alpha_3}{\rightleftarrows}} 4 \right) / \langle \alpha_2\beta_2, \beta_2\alpha_2, \alpha_3\beta_3 \rangle.$$

By direct calculation, we find that $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ \widetilde{\mathcal{H}_4}/L_1)$ is displayed below, where we denote by $\bullet$ τ-tilting modules and by $\circ$ other support τ-tilting (but not τ-tilting) modules.



Hence, $\mathsf{d}_1 = 13$. Similarly, we have $\mathsf{d}_2 = 1$ and $\mathsf{d}_3 = \mathsf{d}_4 = 9$. Therefore, $a_3(\widetilde{\mathcal{H}_4}) = 32$.

Next, we compute the left mutations starting with $\widetilde{\mathcal{H}_4}$ to find all τ-tilting $\widetilde{\mathcal{H}_4}$-modules and the number is 47. We refer to Appendix A.4 for a complete list of τ-tilting $\widetilde{\mathcal{H}_4}$-modules. Besides, the part of $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ \widetilde{\mathcal{H}_4})$ consisting of all τ-tilting $\widetilde{\mathcal{H}_4}$-modules can be obtained. Thus, we have $\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ \mathcal{H}_4 = 96$. $\square$

Theorem 5.2.4. *If the Schur algebra $S(n, r)$ is tame, then it is τ-tilting finite.*

Proof. We have proved in Lemma 5.2.3 that $\mathcal{D}_3$, $\mathcal{D}_4$, $\mathcal{R}_4$ and $\mathcal{H}_4$ are τ-tilting finite. Now, it suffices to make clear that these are all the tame blocks of tame Schur algebras. By Proposition 5.2.1, it is enough to consider $S(2, r)$ for $r = 4, 9, 11$ over $p = 2$, $S(2, r)$ for $r = 9, 10, 11$ and $S(3, r)$ for $r = 7, 8$ over $p = 3$. We have already shown in Example 5.1.7 that the basic algebra of $S(2, 11)$ over $p = 2$ is isomorphic to $\mathcal{D}_3 \oplus \mathcal{A}_2 \oplus \mathbb{F}$. Then, the basic algebra of other tame Schur algebras can be found in [DEMN, Section 5]. We recall the result in [DEMN] as follows.

Let $p = 2$, the basic algebra of $S(2, 4)$ is isomorphic to $\mathcal{D}_3$ and the basic algebra of $S(2, 9)$ is isomorphic to $\mathcal{D}_3 \oplus \mathbb{F} \oplus \mathbb{F}$. Let $p = 3$, the basic algebra of $S(2, 9)$ is isomorphic to $\mathcal{D}_4 \oplus \mathbb{F}$, the basic algebra of $S(2, 10)$ is isomorphic to $\mathcal{D}_4 \oplus \mathbb{F} \oplus \mathbb{F}$ and the basic algebra of $S(2, 11)$ is isomorphic to $\mathcal{D}_4 \oplus \mathcal{A}_2$; the basic algebra of $S(3, 7)$ is isomorphic to $\mathcal{R}_4 \oplus \mathcal{A}_2 \oplus \mathcal{A}_2$ and the basic algebra of $S(3, 8)$ is isomorphic to $\mathcal{R}_4 \oplus \mathcal{H}_4 \oplus \mathcal{A}_2$. $\square$

5.3 Wild Schur algebras

In this section, we consider wild Schur algebras except for a few small cases in $(\star)$. We point out that these four small cases in $(\star)$ have been settled in a separate paper [AW] with Toshitaka Aoki. For the convenience of readers, we will recall the related results without proof in the last section of this chapter.

$$(\star)\ \begin{cases} p = 2, n = 2, r = 8, 17, 19; \\ p = 2, n = 3, r = 4; \\ p = 2, n \geqslant 5, r = 5; \\ p \geqslant 5, n = 2, p^2 \leqslant r \leqslant p^2 + p - 1. \end{cases}$$

Table 5.1: The τ-tilting finite $S(n,r)$ over $p=2$.

r	1	2	3	4	5	6	7	8	9	10	11	12
$S(2,r)$	S	F	S	T	F	W	F	W	T	W	T	W

r	13	14	15	16	17	18	19	20	21	22	23	$\cdots$
$S(2,r)$	W	W	W	W	W	W	W	W	W	W	W	$\cdots$

$n \backslash r$	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
3	S	F	F	W	W	W	W	W	W	W	W	W	W	$\cdots$
4	S	F	F	W	W	W	W	W	W	W	W	W	W	$\cdots$
5	S	F	F	W	W	W	W	W	W	W	W	W	W	$\cdots$
6	S	F	F	W	W	W	W	W	W	W	W	W	W	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

In the rest of this chapter, we will use the decomposition matrix $[S^\lambda : D^\mu]$ of $\mathbb{F}G_r$ given in [Ja] without further notice.

5.3.1 The characteristic $p = 2$

We assume in this subsection that the characteristic of $\mathbb{F}$ is 2. Then, the τ-tilting finiteness for $S(n,r)$ is shown in Table 5.1 and the proof is divided into several propositions as displayed below. Here, the color **purple** means τ-tilting finite, the color **red** means τ-tilting infinite, the capital letter S means semi-simple, the capital letter F means representation-finite, the capital letter T means tame and the capital letter W means wild.

Proposition 5.3.1. *Let $p=2$. Then, $S(2,6)$, $S(2,13)$ and $S(2,15)$ are τ-tilting finite.*

Proof. We consider the Young modules Y^λ for partitions λ with at most two parts.

(1) The part of the decomposition matrix $[S^\lambda : D^\mu]$ for the partitions in the principal block of $\mathbb{F}G_6$ with at most two parts is

$$\begin{array}{c} (6) \\ (5,1) \\ (4,2) \\ (3^2) \end{array} \begin{pmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

and the characters of Young $\mathbb{F}G_6$-modules are given by Lemma 5.1.6 as follows,

$$\begin{aligned}
\mathsf{ch}\ Y^{(6)} &= \chi^{(6)}, \\
\mathsf{ch}\ Y^{(5,1)} &= \chi^{(6)} + \chi^{(5,1)}, \\
\mathsf{ch}\ Y^{(4,2)} &= \chi^{(5,1)} + \chi^{(4,2)}, \\
\mathsf{ch}\ Y^{(3^2)} &= \chi^{(6)} + \chi^{(5,1)} + \chi^{(4,2)} + \chi^{(3^2)}.
\end{aligned}$$

Similar to the method in Example 5.1.7, we compute the radical series of Young modules. Then, one can show that $S_B = \mathsf{End}_{\mathbb{F}G_6}(Y^{(6)} \oplus Y^{(5,1)} \oplus Y^{(4,2)} \oplus Y^{(3^2)})$ is isomorphic to $\mathcal{K}_4 := \mathbb{F}Q/I$ with

$$Q:\ 1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} 3 \underset{\beta_3}{\overset{\alpha_3}{\rightleftarrows}} 4 \ \text{ and } I:\left\langle \begin{matrix} \alpha_1\beta_1, \alpha_2\beta_2, \beta_3\alpha_3, \alpha_1\alpha_2\alpha_3, \beta_3\beta_2\beta_1, \\ \beta_1\alpha_1\alpha_2 - \alpha_2\alpha_3\beta_3, \beta_2\beta_1\alpha_1 - \alpha_3\beta_3\beta_2 \end{matrix} \right\rangle.$$

(See [DEMN, 3.5] for another method to show this.) Since $\beta_1\alpha_1 + \alpha_3\beta_3$ and $\beta_3\beta_2\alpha_2\alpha_3$ are central elements of $\mathcal{K}_4$, we define

$$\widetilde{\mathcal{K}_4} := \mathcal{K}_4 / \langle \beta_1\alpha_1, \alpha_3\beta_3, \beta_3\beta_2\alpha_2\alpha_3 \rangle.$$

We have $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ \mathcal{K}_4) \simeq \mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ \widetilde{\mathcal{K}_4})$ by Proposition 2.3.6. Then, similar to the strategy in Example 2.3.10, we have $a_0(\widetilde{\mathcal{K}_4}) = 1$ and $a_1(\widetilde{\mathcal{K}_4}) = 4$.

Let M be a support τ-tilting $\widetilde{\mathcal{K}_4}$-module with support-rank 2, and with supports e_i and e_j $(i \neq j)$. Then, M becomes a τ-tilting $\widetilde{\mathcal{K}_4}/J$-module with $J :=< 1 - e_i - e_j >$. We denote by $\mathsf{b}_{i,j}$ the number of τ-tilting $\widetilde{\mathcal{K}_4}/J$-modules. Then, it is easy to check that

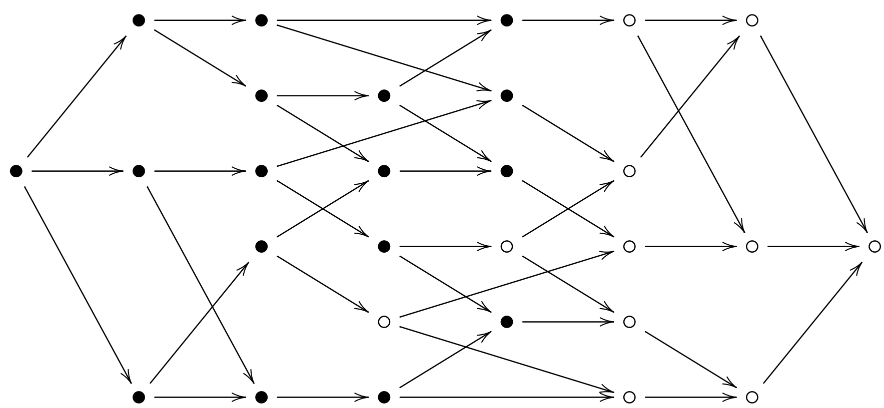
(i,j)	$(1,2)$	$(1,3)$	$(1,4)$	$(2,3)$	$(2,4)$	$(3,4)$
$\mathsf{b}_{i,j}$	3	1	1	3	1	3

This implies that $a_2(\widetilde{\mathcal{K}_4}) = 12$.

Let N be a support τ-tilting $\widetilde{\mathcal{K}_4}$-module with support-rank 3. Then, N becomes a τ-tilting $\widetilde{\mathcal{K}_4}/L_j$-module with $L_j :=< e_j >$, where e_j is the only one non-zero primitive idempotent satisfying $Ne_j = 0$. We denote by d_j the number of τ-tilting $\widetilde{\mathcal{K}_4}/L_j$-modules. If $j = 1$, then

$$\widetilde{\mathcal{K}_4}/L_1 := \mathbb{F}\left(2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} 3 \underset{\beta_3}{\overset{\alpha_3}{\rightleftarrows}} 4 \right) / \langle \alpha_2\beta_2, \beta_3\alpha_3, \alpha_3\beta_3, \beta_3\beta_2\alpha_2\alpha_3 \rangle.$$

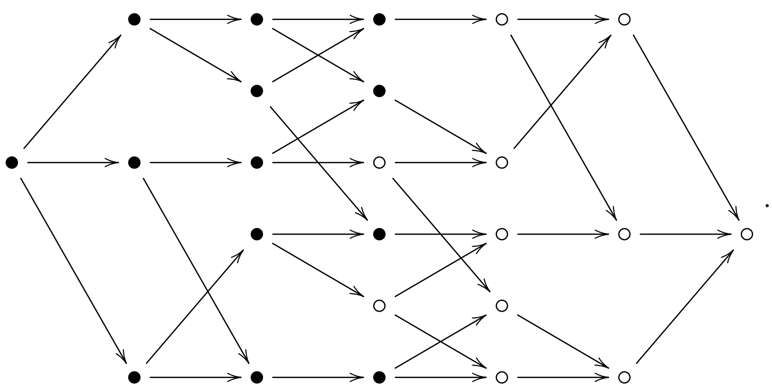
By direct calculation, we can show that the Hasse quiver $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ \widetilde{\mathcal{K}_4}/L_1)$ is displayed below, where we denote by $\bullet$ τ-tilting $\widetilde{\mathcal{K}_4}/L_1$-modules and by $\circ$ other support τ-tilting (but not τ-tilting) $\widetilde{\mathcal{K}_4}/L_1$-modules,



We deduce that $\mathsf{d}_1 = 17$. If $j = 4$, then

$$\widetilde{\mathcal{K}_4}/L_4 := \mathbb{F}\left(1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} 3 \right) / \langle \alpha_1\beta_1, \beta_1\alpha_1, \alpha_2\beta_2 \rangle.$$

Similarly, the Hasse quiver $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ \widetilde{\mathcal{K}_4}/L_4)$ is as follows,



This implies that $\mathsf{d}_4 = 13$. Besides, it is not difficult to see that $\mathsf{d}_2 = \mathsf{d}_3 = 1 \cdot 3 = 3$ by the number $\mathsf{b}_{i,j}$ computed in the previous step. Therefore, $a_3(\widetilde{\mathcal{K}_4}) = 36$.

We compute the left mutations starting with $\widetilde{\mathcal{K}_4}$ to find all τ-tilting $\widetilde{\mathcal{K}_4}$-modules and the number is 83. We refer to Appendix A.5 for a complete list of τ-tilting $\widetilde{\mathcal{K}_4}$-modules. Therefore, we have $\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ \mathcal{K}_4 = 1 + 4 + 12 + 36 + 83 = 136$ and the statement follows from the fact that $\mathcal{K}_4$ is the basic algebra of $S(2,6)$.

(2) The group algebra $\mathbb{F}G_{13}$ contains two blocks, i.e., the principal block B_1 and the block B_2 labeled by 2-core $(2,1)$. The parts of the decomposition matrix $[S^\lambda : D^\mu]$ for the partitions in B_1 and B_2 with at most two parts are

$$B_1 : \begin{array}{c} (13) \\ (11,2) \\ (9,4) \\ (7,6) \end{array} \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 0 & 1 & 1 \end{pmatrix}, \quad B_2 : \begin{array}{c} (12,1) \\ (10,3) \\ (8,5) \end{array} \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 1 & 0 & 1 \end{pmatrix}.$$

We may prove that S_{B_1} is isomorphic to $\mathcal{K}_4$, because the characters of Young modules for $\mathbb{F}G_{13}$ are as follows. (One may compare this with the case of $S(2,6)$.)

$$\begin{aligned} \mathsf{ch}\ Y^{(13)} &= \chi^{(13)}, \\ \mathsf{ch}\ Y^{(11,2)} &= \chi^{(13)} + \chi^{(11,2)}, \\ \mathsf{ch}\ Y^{(9,4)} &= \chi^{(11,2)} + \chi^{(9,4)}, \\ \mathsf{ch}\ Y^{(7,6)} &= \chi^{(13)} + \chi^{(11,2)} + \chi^{(9,4)} + \chi^{(7,6)}. \end{aligned}$$

On the other hand, S_{B_2} is isomorphic to $\mathcal{A}_2 \oplus \mathbb{F}$ by [Er, Proposition 4.1], where $\mathcal{A}_n$ is explained at the start of subsection 5.2.1. Therefore, the basic algebra of $S(2,13)$ is $\mathcal{K}_4 \oplus \mathcal{A}_2 \oplus \mathbb{F}$, which is also τ-tilting finite.

(3) The group algebra $\mathbb{F}G_{15}$ also contains two blocks and the parts of the decomposition matrix $[S^\lambda : D^\mu]$ for the partitions with at most two parts are as follows,

$$\begin{array}{c} (15) \\ (13,2) \\ (11,4) \\ (9,6) \end{array} \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad \begin{array}{c} (14,1) \\ (12,3) \\ (10,5) \\ (8,7) \end{array} \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

After computing the characters of Young $\mathbb{F}G_{15}$-modules by Lemma 5.1.6, we deduce that the basic algebra of $S(2,15)$ is isomorphic to $\mathcal{K}_4 \oplus \mathcal{A}_2 \oplus \mathbb{F} \oplus \mathbb{F}$. $\square$

Now, we look at the case $S(2,8)$. Let B be the principal block of $\mathbb{F}G_8$, the part of the decomposition matrix $[S^\lambda : D^\mu]$ for the partitions in B with at most two parts is

$$
\begin{array}{c}
(8) \\ (7,1) \\ (6,2) \\ (5,3) \\ (4^2)
\end{array}
\begin{pmatrix}
1 & & & \\
1 & 1 & & \\
0 & 1 & 1 & \\
0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1
\end{pmatrix}.
$$

On the other hand, Lemma 5.1.6 implies that the characters of Young $\mathbb{F}G_8$-modules are

$$
\begin{aligned}
\mathsf{ch}\ Y^{(8)} &= \chi^{(8)}, \\
\mathsf{ch}\ Y^{(7,1)} &= \chi^{(8)} + \chi^{(7,1)}, \\
\mathsf{ch}\ Y^{(6,2)} &= \chi^{(8)} + \chi^{(7,1)} + \chi^{(6,2)}, \\
\mathsf{ch}\ Y^{(5,3)} &= \chi^{(6,2)} + \chi^{(5,3)}, \\
\mathsf{ch}\ Y^{(4^2)} &= \chi^{(8)} + \chi^{(7,1)} + \chi^{(6,2)} + \chi^{(5,3)} + \chi^{(4^2)}.
\end{aligned}
$$

It is obvious that $Y^{(8)} = D^{(8)}$ and we may find others as follows.

$$
Y^{(7,1)} = \begin{array}{c} D^{(8)} \\ D^{(7,1)} \\ D^{(8)} \end{array},\ Y^{(6,2)} =
\begin{tikzcd}[arrows=dash]
& D^{(7,1)} \arrow[dl] \arrow[dr] & & & D^{(8)} \arrow[ddl] \\
& & D^{(6,2)} \arrow[dr] & & \\
D^{(8)} & & & D^{(7,1)} &
\end{tikzcd}
,\ Y^{(5,3)} = \begin{array}{c} D^{(6,2)} \\ D^{(7,1)} \\ D^{(5,3)} \\ D^{(7,1)} \\ D^{(6,2)} \end{array},
$$

$$
Y^{(4^2)} =
\begin{tikzcd}[arrows=dash]
& D^{(7,1)} \arrow[d] \arrow[dr] \arrow[ddddl, bend right] & & D^{(8)} \arrow[ddddl, bend left] \\
& D^{(5,3)} \arrow[d] & D^{(6,2)} \arrow[d] & \\
& D^{(7,1)} \arrow[d] & D^{(7,1)} \arrow[d] & \\
& D^{(6,2)} \arrow[dr] & D^{(5,3)} \arrow[d] & \\
D^{(8)} & & D^{(7,1)} &
\end{tikzcd}.
$$

Note that the dimension of $\mathsf{Hom}_{\mathbb{F}G_8}(Y^\lambda, Y^\mu)$ between two Young modules Y^λ, Y^μ is equal to the inner product $(\mathsf{ch}\ Y^\lambda, \mathsf{ch}\ Y^\mu)$. By direct calculation, we conclude that $S_B = \mathsf{End}_{\mathbb{F}G_8}(Y^{(8)} \oplus Y^{(7,1)} \oplus Y^{(6,2)} \oplus Y^{(5,3)} \oplus Y^{(4^2)})$ is isomorphic to $\mathcal{L}_5 := \mathbb{F}Q/I$ with

$$
Q:\ \begin{tikzcd}
(5,3) \arrow[r, shift left, "\alpha_1"] & (4^2) \arrow[l, shift left, "\beta_1"] \arrow[r, shift left, "\alpha_2"] \arrow[d, shift left, "\alpha_4"] & (6,2) \arrow[l, shift left, "\beta_2"] \arrow[r, shift left, "\alpha_3"] & (7,1) \arrow[l, shift left, "\beta_3"] \\
& (8) \arrow[u, shift left, "\beta_4"] & &
\end{tikzcd}
\ \text{and}
$$

$$
I: \left\langle \begin{array}{c}
\alpha_1\beta_1, \alpha_1\alpha_4, \beta_3\alpha_3, \beta_2\alpha_2, \beta_4\alpha_4, \beta_4\beta_1, \beta_4\alpha_2\beta_2, \alpha_1\alpha_2\alpha_3, \alpha_2\beta_2\alpha_4, \beta_3\beta_2\beta_1, \\
\beta_1\alpha_1\alpha_2 - \alpha_2\alpha_3\beta_3, \beta_2\beta_1\alpha_1 - \alpha_3\beta_3\beta_2, \alpha_2\beta_2\beta_1\alpha_1 - \beta_1\alpha_1\alpha_2\beta_2
\end{array} \right\rangle.
$$

Here, we replace each vertex in the quiver of S_B by the partition λ associated with the Young module Y^λ. We refer to the last subsection for the τ-tilting finiteness of $\mathcal{L}_5$.

Proposition 5.3.2. *Let $p = 2$. Then, the wild Schur algebras $S(2, 17)$ and $S(2, 19)$ are τ-tilting finite if and only if $S(2, 8)$ is τ-tilting finite.*

Proof. We show that the basic algebra of $S(2, 17)$ is isomorphic to $\mathcal{L}_5 \oplus \mathcal{A}_2 \oplus \mathbb{F} \oplus \mathbb{F}$. The blocks of $\mathbb{F}G_{17}$ and the parts of the decomposition matrix $[S^\lambda : D^\mu]$ for the partitions with at most two parts are as follows,

$$\begin{array}{c} (17) \\ (15,2) \\ (13,4) \\ (11,6) \\ (9,8) \end{array} \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 0 & 1 & 1 & & \\ 0 & 1 & 1 & 1 & \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}, \quad \begin{array}{c} (16,1) \\ (14,3) \\ (12,5) \\ (10,7) \end{array} \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

In order to identity $\mathcal{L}_5$, it suffices to check the characters of Young $\mathbb{F}G_{17}$-modules:

$$\begin{aligned} \mathsf{ch}\ Y^{(17)} &= \chi^{(17)}, \\ \mathsf{ch}\ Y^{(15,2)} &= \chi^{(17)} + \chi^{(15,2)}, \\ \mathsf{ch}\ Y^{(13,4)} &= \chi^{(17)} + \chi^{(15,2)} + \chi^{(13,4)}, \\ \mathsf{ch}\ Y^{(11,6)} &= \chi^{(13,4)} + \chi^{(11,6)}, \\ \mathsf{ch}\ Y^{(9,8)} &= \chi^{(17)} + \chi^{(15,2)} + \chi^{(13,4)} + \chi^{(11,6)} + \chi^{(9,8)}. \end{aligned}$$

For the case $S(2, 19)$, the blocks of $\mathbb{F}G_{19}$ and the parts of the decomposition matrix $[S^\lambda : D^\mu]$ for the partitions with at most two parts are

$$\begin{array}{c} (19) \\ (17,2) \\ (15,4) \\ (13,6) \\ (11,8) \end{array} \begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ 1 & 0 & 1 & & \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}, \quad \begin{array}{c} (18,1) \\ (16,3) \\ (14,5) \\ (12,7) \\ (10,9) \end{array} \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 0 & 1 & 1 & & \\ 0 & 1 & 1 & 1 & \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Also, by Lemma 5.1.6, we have

$$\begin{aligned} \mathsf{ch}\ Y^{(19)} &= \chi^{(19)},\ \mathsf{ch}\ Y^{(15,4)} = \chi^{(19)} + \chi^{(15,4)}, \\ \mathsf{ch}\ Y^{(11,8)} &= \chi^{(19)} + \chi^{(15,4)} + \chi^{(11,8)}; \\ \mathsf{ch}\ Y^{(18,1)} &= \chi^{(18,1)},\ \mathsf{ch}\ Y^{(16,3)} = \chi^{(18,1)} + \chi^{(16,3)}, \\ \mathsf{ch}\ Y^{(14,5)} &= \chi^{(18,1)} + \chi^{(16,3)} + \chi^{(14,5)},\ \mathsf{ch}\ Y^{(12,7)} = \chi^{(14,5)} + \chi^{(12,7)}, \\ \mathsf{ch}\ Y^{(10,9)} &= \chi^{(18,1)} + \chi^{(16,3)} + \chi^{(14,5)} + \chi^{(12,7)} + \chi^{(10,9)}. \end{aligned}$$

Similar to the above, the basic algebra of $S(2, 19)$ is isomorphic to $\mathcal{L}_5 \oplus \mathcal{D}_3 \oplus \mathbb{F} \oplus \mathbb{F}$, where $\mathcal{D}_3$ is a τ-tilting finite algebra defined in subsection 5.2.2. $\square$

Proposition 5.3.3. *Let $p = 2$.*

(1) If r is even, then $S(2, r)$ is τ-tilting infinite for any $r \geqslant 10$.

(2) If r is odd, then $S(2, r)$ is τ-tilting infinite for any $r \geqslant 21$.

Proof. We denote by $\overline{S(2,r)}$ the basic algebra of $S(2,r)$ and we use Lemma 5.1.11 to determine the quiver of $\overline{S(2,r)}$. When we display the quiver of $\overline{S(2,r)}$, we usually replace each vertex by the partition λ associated with Y^λ. Then, the quiver of $\overline{S(2,10)}$ is

$$\begin{tikzcd} & & (6,4) \arrow[r, shift left] \arrow[d, shift left] & (10) \arrow[l, shift left] \arrow[d, shift left] \\ (8,2) \arrow[r, shift left] & (7,3) \arrow[l, shift left] \arrow[r, shift left] & (5^2) \arrow[l, shift left] \arrow[u, shift left] \arrow[r, shift left] & (9,1) \arrow[l, shift left] \arrow[u, shift left] \end{tikzcd},$$

and the quiver of $\overline{S(2,21)}$ is

$$\begin{tikzcd} (17,4) \arrow[d, shift left] & (13,8) \arrow[r, shift left] \arrow[d, shift left] & (21) \arrow[l, shift left] \arrow[d, shift left] & (14,7) & & (18,3) \\ (15,6) \arrow[u, shift left] \arrow[r, shift left] & (11,10) \arrow[l, shift left] \arrow[u, shift left] \arrow[r, shift left] & (19,2) \arrow[l, shift left] \arrow[u, shift left] & (20,1) \arrow[r, shift left] & (12,9) \arrow[l, shift left] \arrow[r, shift left] & (16,5) \arrow[l, shift left] \end{tikzcd}.$$

Now, it is enough to say that $S(2,10)$ and $S(2,21)$ are τ-tilting infinite by Lemma 5.1.10. Hence, the statement follows from Lemma 5.1.9. □

Proposition 5.3.4. *Let $p=2$. Then,*

(1) the wild Schur algebra $S(3,5)$ is τ-tilting finite.

(2) the wild Schur algebra $S(n,r)$ is τ-tilting infinite for any $n \geqslant 3$ and $r \geqslant 6$.

Proof. We consider the Young modules Y^λ for partitions λ with at most three parts. Then, Specht modules S^μ in the Specht filtration of Y^λ and composition factors D^μ which appear in Y^λ are also corresponding to the partitions with at most three parts.

(1) We show that the basic algebra of $S(3,5)$ is τ-tilting finite. The group algebra $\mathbb{F}G_5$ contains only two blocks, i.e. the principal block B_1 and the block B_2 labeled by 2-core $(2,1)$. The parts of the decomposition matrix $[S^\lambda : D^\mu]$ for the partitions in B_1 and B_2 with at most three parts are as follows,

$$B_1 : \begin{array}{c} (5) \\ (3,2) \\ (3,1^2) \\ (2^2,1) \end{array} \begin{pmatrix} 1 & \\ 1 & 1 \\ 2 & 1 \\ 1 & 1 \end{pmatrix}, \ B_2 : (4,1) \begin{pmatrix} 1 \end{pmatrix}.$$

Combining with [Er, Proposition 5.8], the basic algebra of $S(3,5)$ is isomorphic to $\mathcal{U}_4 \oplus \mathbb{F}$, where $\mathcal{U}_4 := \mathbb{F}Q/I$ is presented by

$$Q: \ 1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} 3 \underset{\beta_3}{\overset{\alpha_3}{\rightleftarrows}} 4 \ \text{ and } I : \left\langle \alpha_1\beta_1, \alpha_2\beta_2, \alpha_1\alpha_2\alpha_3, \beta_3\beta_2\beta_1, \alpha_3\beta_3 - \beta_2\alpha_2 \right\rangle.$$

Since $\beta_2\alpha_2 + \beta_3\alpha_3$ and $\beta_2\beta_1\alpha_1\alpha_2$ are non-trivial central elements of $\mathcal{U}_4$, the τ-tilting finiteness of $\mathcal{U}_4$ is the same as $\widetilde{\mathcal{U}_4} := \mathcal{U}_4 / \langle \beta_2\alpha_2, \beta_3\alpha_3, \beta_2\beta_1\alpha_1\alpha_2 \rangle$ by Proposition 2.3.6. Then, similar to the strategy in Example 2.3.10, we have $a_0(\widetilde{\mathcal{U}_4}) = 1$ and $a_1(\widetilde{\mathcal{U}_4}) = 4$.

Let M be a support τ-tilting $\widetilde{\mathcal{U}_4}$-module with support-rank 2, and with supports e_i and e_j $(i \neq j)$. Then, M becomes a τ-tilting $\widetilde{\mathcal{U}_4}/J$-module with $J :=< 1 - e_i - e_j >$. We denote by $\mathsf{b}_{i,j}$ the number of τ-tilting $\widetilde{\mathcal{U}_4}/J$-modules. Then, it is easy to check that

$$
\begin{array}{c|cccccc}
(i,j) & (1,2) & (1,3) & (1,4) & (2,3) & (2,4) & (3,4) \\
\hline
\mathsf{b}_{i,j} & 3 & 1 & 1 & 3 & 1 & 3
\end{array}.
$$

This implies that $a_2(\widetilde{\mathcal{U}_4}) = 12$.

Let N be a support τ-tilting $\widetilde{\mathcal{U}_4}$-module with support-rank 3. Then, N becomes a τ-tilting $\widetilde{\mathcal{U}_4}/L_j$-module with $L_j :=< e_j >$, where e_j is the only one non-zero primitive idempotent satisfying $Ne_j = 0$. We denote by d_j the number of τ-tilting $\widetilde{\mathcal{U}_4}/L_j$-modules. Similar to the case $\widetilde{\mathcal{K}_4}$ in Proposition 5.3.1, we have $\mathsf{d}_1 = 13$, $\mathsf{d}_2 = \mathsf{d}_3 = 3$ and $\mathsf{d}_4 = 17$. Therefore, $a_3(\widetilde{\mathcal{U}_4}) = 36$.

Finally, we compute the left mutations starting with $\widetilde{\mathcal{U}_4}$ to find all τ-tilting $\widetilde{\mathcal{U}_4}$-modules and the number is 83. Since this number can be verified by the String Applet [Geu], we omit the detailed computation. Therefore, we have $\#\mathsf{s\tau\text{-}tilt}\,\mathcal{U}_4 = 1+4+12+36+83 = 136$.

(2) We shall show that $S(3,6)$, $S(3,7)$ and $S(3,8)$ are τ-tilting infinite. Then, the statement follows from Lemma 5.1.8 and Lemma 5.1.9. As we are already familiar with the strategy of determining the radical series of Young modules and the basic algebras of Schur algebras, we may leave this heavy work to a computer and some mathematicians indeed did. Here, we refer to Carlson and Matthews's program [CM].

(2.1) Let B be the principal block of $\mathbb{F}G_6$, the part of the decomposition matrix $[S^\lambda : D^\mu]$ for the partitions in B with at most three parts is of the form

$$
\begin{array}{c}
(6) \\ (5,1) \\ (4,2) \\ (4,1^2) \\ (3^2) \\ (2^3)
\end{array}
\begin{pmatrix}
1 & & \\
1 & 1 & \\
1 & 1 & 1 \\
2 & 1 & 1 \\
1 & 0 & 1 \\
1 & 0 & 1
\end{pmatrix}.
$$

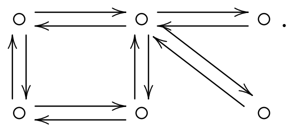
Then, the quiver of $S_B = \mathsf{End}_{\mathbb{F}G_6}(\bigoplus_{\lambda\in B\cap\Omega(3,6)} Y^\lambda)$ is as follows,

$$
\begin{tikzcd}
\circ \arrow[r, shift left] \arrow[d, shift left] & \circ \arrow[l, shift left] \arrow[r, shift left] \arrow[d, shift left] & \circ \arrow[l, shift left] \arrow[d, shift left] \\
\circ \arrow[u, shift left] \arrow[r, shift left] & \circ \arrow[u, shift left] \arrow[l, shift left] \arrow[r, shift left] & \circ \arrow[u, shift left] \arrow[l, shift left]
\end{tikzcd}.
$$

(2.2) Let B be the principal block of $\mathbb{F}G_7$, the part of the decomposition matrix $[S^\lambda : D^\mu]$ for the partitions in B with at most three parts is of the form

$$
\begin{array}{c}
(7) \\ (5,2) \\ (4,2,1) \\ (5,1^2) \\ (3^2,1) \\ (3,2^2)
\end{array}
\begin{pmatrix}
1 & & \\
0 & 1 & \\
1 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 1 \\
1 & 0 & 1
\end{pmatrix}.
$$

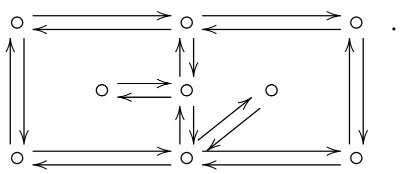
Then, the quiver of $S_B = \mathsf{End}_{\mathbb{F}G_7}(\bigoplus_{\lambda\in B\cap\Omega(3,7)} Y^\lambda)$ is as follows,



(2.3) Let B be the principal block of $\mathbb{F}G_8$, the part of the decomposition matrix $[S^\lambda : D^\mu]$ for the partitions in B with at most three parts is of the form

$$\begin{array}{c} (8) \\ (7,1) \\ (6,2) \\ (5,3) \\ (4,3,1) \\ (4^2) \\ (6,1^2) \\ (4,2^2) \\ (3^2,2) \end{array} \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 0 & 1 & 1 & & \\ 0 & 1 & 1 & 1 & \\ 2 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then, the quiver of $S_B = \mathsf{End}_{\mathbb{F}G_8}(\bigoplus_{\lambda \in B\cap\Omega(3,8)} Y^\lambda)$ is as follows,



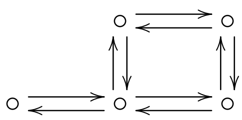
By Lemma 5.1.10, we conclude that $S(3,6)$, $S(3,7)$ and $S(3,8)$ are τ-tilting infinite. $\square$

Corollary 5.3.5. *Let $p = 2$. The wild Schur algebra $S(4,5)$ is τ-tilting finite.*

Proof. We consider the Young modules Y^λ for partitions λ of 5 with at most four parts. Note that $S(3,5)$ is an idempotent truncation of $S(4,5)$ as we mentioned in Lemma 5.1.8. Compared with the case $S(3,5)$, the case $S(4,5)$ has only one additional partition $(2,1^3)$ which appears in the block of $\mathbb{F}G_5$ labeled by 2-core (2,1). Then, the basic algebra of $S(4,5)$ is isomorphic to $\mathcal{U}_4 \oplus \mathcal{A}_2$ based on the result on $S(3,5)$. $\square$

Proposition 5.3.6. *Let $p = 2$. The algebra $S(n,4)$ is τ-tilting infinite for any $n \geqslant 4$.*

Proof. By our strategy in Section 5.1, one can see that $S(n,4)$ with $n \geqslant 5$ is always Morita equivalent to $S(4,4)$. So it is enough to show that $S(4,4)$ is τ-tilting infinite. In fact, the quiver of the basic algebra of $S(4,4)$ displayed below implies our statement.



This quiver has been given by Xi in [Xi]. $\square$

Hence, we have already determined the τ-tilting finiteness of wild Schur algebras over $p = 2$, except for $S(2,r)$ with $r = 8, 17, 19$, $S(3,4)$ and $S(n,5)$ with $n \geqslant 5$.

Table 5.2: The τ-tilting finite $S(n,r)$ over $p=3$.

$n \backslash r$	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
2	S	S	F	F	F	F	F	F	T	T	T	W	W	$\cdots$
3	S	S	F	F	F	W	T	T	W	W	W	W	W	$\cdots$
4	S	S	F	F	F	W	W	W	W	W	W	W	W	$\cdots$
5	S	S	F	F	F	W	W	W	W	W	W	W	W	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Remark 5.3.7. We recall from [DEMN, 3.6] that the basic algebra of $S(3,4)$ over $p=2$ is presented by the bound quiver algebra $\mathcal{M}_4 := \mathbb{F}Q/I$ with

$$Q: \begin{tikzcd} \circ \arrow[r, shift left, "\alpha_1"] & \circ \arrow[l, shift left, "\beta_1"] \arrow[r, shift left, "\alpha_2"] \arrow[d, shift left, "\alpha_3"] & \circ \arrow[l, shift left, "\beta_2"] \\ & \circ \arrow[u, shift left, "\beta_3"] & \end{tikzcd} \quad \text{and } I: \left\langle \begin{array}{c} \alpha_1\beta_1, \beta_3\alpha_3, \alpha_1\alpha_2, \beta_2\beta_1, \\ \alpha_1\alpha_3\beta_3, \alpha_3\beta_3\beta_1, \beta_1\alpha_1 - \alpha_2\beta_2 \end{array} \right\rangle.$$

Remark 5.3.8. Let $p=2$. The wild Schur algebra $S(n,5)$ with $n \geqslant 6$ is always Morita equivalent to $S(5,5)$. Moreover, the basic algebra of $S(5,5)$ is isomorphic to $\mathcal{N}_5 \oplus \mathcal{A}_2$ following [Xi, Proposition 3.8], where $\mathcal{N}_5 := \mathbb{F}Q/I$ is presented by

$$Q: \begin{tikzcd} \circ \arrow[r, shift left, "\alpha_1"] & \circ \arrow[l, shift left, "\beta_1"] \arrow[r, shift left, "\alpha_2"] & \circ \arrow[l, shift left, "\beta_2"] \arrow[r, shift left, "\alpha_3"] & \circ \arrow[l, shift left, "\beta_3"] \arrow[r, shift left, "\alpha_4"] & \circ \arrow[l, shift left, "\beta_4"] \end{tikzcd} \text{ with}$$

$$I: \left\langle \begin{array}{c} \alpha_1\beta_1, \alpha_2\beta_2, \alpha_3\beta_3, \beta_4\alpha_4, \alpha_1\alpha_2\alpha_3\alpha_4, \beta_4\beta_3\beta_2\beta_1, \beta_2\alpha_2 - \alpha_3\alpha_4\beta_4\beta_3, \\ \alpha_2\alpha_3\alpha_4\beta_4 - \beta_1\alpha_1\alpha_2\alpha_3, \beta_3\beta_2\beta_1\alpha_1 - \alpha_4\beta_4\beta_3\beta_2 \end{array} \right\rangle.$$

5.3.2 The characteristic $p=3$

We assume in this subsection that the characteristic of $\mathbb{F}$ is 3. Then, the τ-tilting finiteness for $S(n,r)$ is shown in Table 5.2 and the proof is divided into the propositions displayed below. Here, we use the same conventions with Table 5.1.

Proposition 5.3.9. *Let $p=3$. Then, $S(2,r)$ is τ-tilting infinite for any $r \geqslant 12$.*

Proof. We show that both $S(2,12)$ and $S(2,13)$ are τ-tilting infinite and the statement follows from Lemma 5.1.9. In fact, let B be the principal block of $\mathbb{F}G_{12}$ and the quiver of $S_B = \mathsf{End}_{\mathbb{F}G_{12}}(\bigoplus_{\lambda \in B \cap \Omega(2,12)} Y^\lambda)$ following Lemma 5.1.11 is

$$\begin{tikzcd} & (11,1) \arrow[r, shift left] \arrow[d, shift left] & (12) \arrow[l, shift left] \arrow[d, shift left] \\ (9,3) \arrow[r, shift left] & (8,4) \arrow[l, shift left] \arrow[u, shift left] \arrow[r, shift left] & (6^2) \arrow[l, shift left] \arrow[u, shift left] \end{tikzcd},$$

where we replace each vertex by the partition λ associated with Y^λ. Thus, $S(2,12)$ is τ-tilting infinite by Lemma 5.1.10. One can check that $S(2,13)$ also contains a τ-tilting infinite subquiver as shown above. $\square$

In the following, we refer to [CM] for the quiver of S_B without further notice.

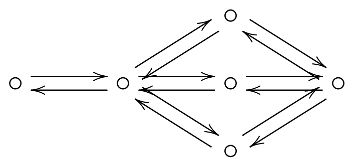
Proposition 5.3.10. *Let $p = 3$. Then, $S(3,6)$ and $S(3,r)$ with $r \geqslant 9$ are τ-tilting infinite.*

Proof. According to Lemma 5.1.9, it suffices to show that $S(3,6)$, $S(3,10)$ and $S(3,11)$ are τ-tilting infinite.

(1) Let B be the principal block of $\mathbb{F}G_6$, the part of the decomposition matrix $[S^\lambda : D^\mu]$ for the partitions in B with at most three parts is of the form

$$\begin{array}{c} (6) \\ (5,1) \\ (4,1^2) \\ (3^2) \\ (3,2,1) \\ (2^3) \end{array} \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 0 & 1 & 1 & & \\ 0 & 1 & 0 & 1 & \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

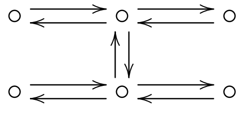
Then, the quiver of $S_B = \mathsf{End}_{\mathbb{F}G_6}(\bigoplus_{\lambda \in B \cap \Omega(3,6)} Y^\lambda)$ is as follows.



(2) Let B_1 be the principal block of $\mathbb{F}G_{10}$ and B_2 the block of $\mathbb{F}G_{11}$ labeled by 3-core (1^2). Then, the parts of the decomposition matrix $[S^\lambda : D^\mu]$ for the partitions in B_1 and B_2 with at most three parts are of the form

$$B_1 : \begin{array}{c} (10) \\ (8,2) \\ (7,3) \\ (7,2,1) \\ (5^2) \\ (4,3^2) \end{array} \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 0 & 1 & 1 & & & \\ 1 & 1 & 1 & 1 & & \\ 0 & 0 & 1 & 0 & 1 & \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad B_2 : \begin{array}{c} (10,1) \\ (9,2) \\ (7,4) \\ (7,2^2) \\ (6,5) \\ (4^2,3) \end{array} \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 0 & 1 & 1 & & & \\ 1 & 1 & 1 & 1 & & \\ 0 & 0 & 1 & 0 & 1 & \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Then, both the quivers of S_{B_1} and S_{B_2} are as follows.



By Lemma 5.1.10, the above two cases are τ-tilting infinite quivers. $\square$

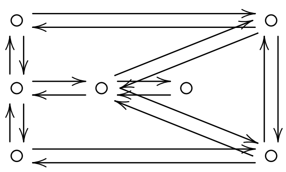
Proposition 5.3.11. *Let $p = 3$. The wild Schur algebra $S(n,r)$ is τ-tilting infinite for any $n \geqslant 4$ and $r \geqslant 6$.*

Proof. Based on the result of $S(3,r)$, Lemma 5.1.8 and Lemma 5.1.9, it suffices to show that $S(4,7)$ and $S(4,8)$ are τ-tilting infinite.

(1) Let B be the principal block of $\mathbb{F}G_7$, the part of the decomposition matrix $[S^\lambda : D^\mu]$ for the partitions in B with at most four parts is of the form

$$\begin{array}{c} (7) \\ (5,2) \\ (4,3) \\ (4,2,1) \\ (3,2,1^2) \\ (4,1^3) \\ (2^3,1) \end{array} \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 0 & 1 & 1 & & \\ 1 & 1 & 1 & 1 & \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

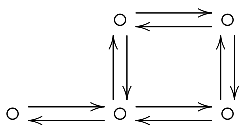
Then, the quiver of $S_B = \mathsf{End}_{\mathbb{F}G_7}(\bigoplus_{\lambda \in B \cap \Omega(4,7)} Y^\lambda)$ is as follows.



(2) Let B be the block of $\mathbb{F}G_8$ labeled by 3-core (1^2), the part of the decomposition matrix $[S^\lambda : D^\mu]$ for the partitions in B with at most four parts is of the form

$$\begin{array}{c} (7,1) \\ (6,2) \\ (4^2) \\ (4,2^2) \\ (3,2^2,1) \end{array} \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 0 & 1 & 1 & & \\ 1 & 1 & 1 & 1 & \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Then, the quiver of $S_B = \mathsf{End}_{\mathbb{F}G_8}(\bigoplus_{\lambda \in B \cap \Omega(4,8)} Y^\lambda)$ is as follows.



Obviously, $S(4,7)$ and $S(4,8)$ are τ-tilting infinite. $\square$

Hence, we have determined the τ-tilting finiteness for all the cases over $p = 3$.

5.3.3 The characteristic $p \geqslant 5$

The situation on $p \geqslant 5$ is much easier than the situation on $p = 2, 3$. As shown in Proposition 5.2.1, tame Schur algebras do not appear in this case. Then, the τ-tilting finiteness for $S(n, r)$ is shown in Table 5.3 and the proof is divided into two propositions. Here, we use the same conventions with Table 5.1.

Proposition 5.3.12. *Let $p \geqslant 5$. The algebra $S(2, r)$ is τ-tilting infinite for any $r \geqslant p^2 + p$.*

Proof. It suffices to consider $S(2, p^2+p)$ and $S(2, p^2+p+1)$ following Lemma 5.1.9. To show the τ-tilting finiteness of $S(2, p^2+p)$, we choose four partitions

$$(p^2+p), (p^2+p-1, 1), (p^2-p, 2p), (p^2-1, p+1),$$

Table 5.3: The τ-tilting finite $S(n,r)$ over $p \geqslant 5$.

n \ r	$1 \sim p$	$p+1 \sim 2p-1$	$2p \sim p^2-1$	$p^2 \sim p^2+p-1$	$p^2+p \sim \infty$
2	S	F	F	W	W
3	S	F	W	W	W
4	S	F	W	W	W
5	S	F	W	W	W
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

which are contained in the principal block B of $\mathbb{F}G_{p^2+p}$. By Lemma 5.1.11, one may construct the following subquiver in the quiver of S_B.

$$\begin{tikzcd}
v^{(p^2+p)} \arrow[r, shift left] \arrow[d, shift left] & v^{(p^2+p-1,1)} \arrow[l, shift left] \arrow[d, shift left] \\
v^{(p^2-p,2p)} \arrow[r, shift left] \arrow[u, shift left] & v^{(p^2-1,p+1)} \arrow[l, shift left] \arrow[u, shift left]
\end{tikzcd}$$

This is just the τ-tilting infinite quiver Q_1 and therefore, $S(2,p^2+p)$ is τ-tilting infinite.

Moreover, we can show that $S(2,p^2+p+1)$ contains the τ-tilting infinite quiver Q_1 as a subquiver if we choose $(p^2+p+1), (p^2+p-1,2), (p^2-p+1,2p)$ and $(p^2-1,p+2)$. $\square$

Proposition 5.3.13. *Let $p \geqslant 5$. The wild Schur algebra $S(n,r)$ is τ-tilting infinite for any $n \geqslant 3$ and $r \geqslant 2p$.*

Proof. It suffices to consider $S(3,r)$ for $r = 2p+x$ with $0 \leqslant x \leqslant 2$. Let B be the principal block of $\mathbb{F}G_r$. Then, the part of the decomposition matrix $[S^\lambda : D^\mu]$ for the partitions in B with at most three parts is of the form

$$\begin{array}{c} (2p+x) \\ (2p-1,1+x) \\ (p+x,p) \\ (2p-2,1+x,1) \\ (p+x,p-1,1) \\ ((p-1)^2,2+x) \end{array} \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 0 & 1 & 1 & & & \\ 0 & 1 & 0 & 1 & & \\ 1 & 1 & 1 & 1 & 1 & \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

We recall from [Er, Proposition 5.3.1] that the quiver of S_B is

$$\begin{tikzcd}
& & \circ \arrow[dl, shift left] \arrow[dr, shift left] & \\
\circ \arrow[r, shift left] & \circ \arrow[l, shift left] \arrow[ur, shift left] \arrow[r, shift left] \arrow[dr, shift left] & \circ \arrow[l, shift left] \arrow[r, shift left] & \circ \arrow[l, shift left] \arrow[ul, shift left] \arrow[dl, shift left] \\
& & \circ \arrow[ul, shift left] \arrow[ur, shift left] &
\end{tikzcd}.$$

Then, the statement follows from Lemma 5.1.8, 5.1.9 and 5.1.10. $\square$

Hence, we have already determined the τ-tilting finiteness of wild Schur algebras over $p \geqslant 5$, except for $S(2,r)$ with $p^2 \leqslant r \leqslant p^2+p-1$.

5.3.4 The remaining cases

In this subsection, we present the τ-tilting finiteness of the remaining cases. We refer to [AW] for details and proofs.

(1) $p = 2, n = 2, r = 8, 17, 19$. It is proved in Proposition 5.3.2 that the basic algebra of $S(2, 8)$ is isomorphic to $\mathcal{L}_5$, the basic algebra of $S(2, 17)$ is isomorphic to $\mathbb{F} \oplus \mathbb{F} \oplus \mathcal{A}_2 \oplus \mathcal{L}_5$ and the basic algebra of $S(2, 19)$ is isomorphic to $\mathbb{F} \oplus \mathbb{F} \oplus \mathcal{D}_3 \oplus \mathcal{L}_5$. Since $\mathbb{F}, \mathcal{A}_2$ and $\mathcal{D}_3$ are τ-tilting finite, the problem in this case is to determine the τ-tilting finiteness of $\mathcal{L}_5$.

(2) $p = 2, n = 3, r = 4$. It is claimed in Remark 5.3.7 that the basic algebra of $S(3, 4)$ is isomorphic to $\mathcal{M}_4$.

(3) $p = 2, n \geqslant 5, r = 5$. It is claimed in Remark 5.3.8 that the basic algebra of $S(n, 5)$ is Morita equivalent to $\mathcal{N}_5 \oplus \mathcal{A}_2$.

(4) $p \geqslant 5, n = 2, p^2 \leqslant r \leqslant p^2 + p - 1$. We recall from [AW] that the Schur algebra $S(2, r)$ with $p^2 \leqslant r \leqslant p^2 + p - 1$ contains only $\mathbb{F}$, $\mathcal{A}_n$ with $2 \leqslant n \leqslant p$ and $\mathcal{D}_{p+1}$ as block algebras, where $\mathcal{D}_m := \mathbb{F}Q/I$ $(m \geqslant 5)$ is presented by

$$Q: \begin{tikzcd} 1 \arrow[dr, shift left, "\alpha_1"] & & & & & \\ & 3 \arrow[ul, shift left, "\beta_1"] \arrow[dl, shift left, "\beta_2"] \arrow[r, shift left, "\mu_3"] & 4 \arrow[l, shift left, "\nu_3"] \arrow[r, shift left, "\mu_4"] & \cdots \arrow[l, shift left, "\nu_4"] \arrow[r, shift left, "\mu_{m-2}"] & m-1 \arrow[l, shift left, "\nu_{m-2}"] \arrow[r, shift left, "\mu_{m-1}"] & m \arrow[l, shift left, "\nu_{m-1}"] \\ 2 \arrow[ur, shift left, "\alpha_2"] & & & & & \end{tikzcd} \text{ with}$$

$$I: \left\langle \begin{array}{c} \alpha_1\beta_1, \alpha_2\beta_2, \alpha_1\beta_2\alpha_2, \beta_2\alpha_2\beta_1, \alpha_1\mu_3, \alpha_2\mu_3, \nu_3\beta_1, \nu_3\beta_2, \\ \mu_3\nu_3 - \beta_2\alpha_2, \mu_i\mu_{i+1}, \nu_{i+1}\nu_i, \nu_i\mu_i - \mu_{i+1}\nu_{i+1}, 3 \leqslant i \leqslant m-2 \end{array} \right\rangle.$$

Note that $\mathcal{D}_3$ and $\mathcal{D}_4$ are defined in Section 5.2. Since $\mathbb{F}$ and $\mathcal{A}_n$ are τ-tilting finite, the problem in this case is to determine the τ-tilting finiteness of $\mathcal{D}_{p+1}$.

Finally, we have

Theorem 5.3.14 ([AW]). *Let $S(n, r)$ be the Schur algebra over $\mathbb{F}$.*

(1) If $p = 2$, then $S(2, 8)$, $S(2, 17)$ and $S(2, 19)$ are τ-tilting finite.

(2) If $p = 2$, then $S(3, 4)$ is τ-tilting finite.

(3) If $p = 2$, then $S(n, 5)$ is τ-tilting infinite for any $n \geqslant 5$.

(4) If $p \geqslant 5$, then $S(2, r)$ is τ-tilting finite for any $p^2 \leqslant r \leqslant p^2 + p - 1$.

Appendix A

A.1 Table T and Table W

For the algebras T_i and W_i in Table T and Table W, respectively, we mean the bound quiver algebras with an admissible ideal generated by the relation (i).

Table T

$$\begin{tikzcd} \circ \arrow[r, shift left=3, "\mu_1,\mu_2"] \arrow[r, shift left=1] & \circ \arrow[l, shift left=1] \arrow[l, shift left=3, "\nu_1,\nu_2"] \end{tikzcd}$$

(1) $\nu_1\mu_1 = \nu_2\mu_2 = (\ell_1\mu_1 + \ell_2\mu_2)(k_1\nu_1 + k_2\nu_2) = (\ell_3\mu_1 + \ell_4\mu_2)(k_3\nu_1 + k_4\nu_2) = 0$,
where $k_1, k_2, k_3, k_4, \ell_1, \ell_2, \ell_3, \ell_4 \in K$ and $k_1k_4 \neq k_2k_3, \ell_1\ell_4 \neq \ell_2\ell_3$.

$$\begin{tikzcd} \circ \arrow[loop left, "\alpha"] \arrow[r, "\mu"] & \circ \end{tikzcd}$$

(2) $\alpha^6 = \alpha^2\mu = 0$;

$$\begin{tikzcd} \circ \arrow[loop left, "\alpha"] \arrow[r, "\mu"] & \circ \arrow[loop right, "\beta"] \end{tikzcd}$$

(3) $\alpha^2 = \beta^2 = 0$;

(4) $\alpha^2 = \beta^n = \mu\beta = 0,\ 2 \leqslant n \in \mathbb{N}$;

(5) $\alpha^m = \beta^n = \alpha\mu = \mu\beta = 0,\ 2 \leqslant m, n \in \mathbb{N}$;

(6) $\alpha^2 = \beta^3 = 0, \alpha\mu = \mu\beta^2$;

(7) $\alpha^3 = \beta^6 = 0, \alpha\mu = \mu\beta$;

(8) $\alpha^4 = \beta^4 = 0, \alpha\mu = \mu\beta$;

$$\begin{tikzcd} \circ \arrow[loop left, "\alpha"] \arrow[r, shift left, "\mu"] & \circ \arrow[l, shift left, "\nu"] \end{tikzcd}$$

(9) $\alpha^2 = \mu\nu\mu = \nu\mu\nu = (\nu\alpha\mu)^n = 0$, $1 \leqslant n \in \mathbb{N}$;

(10) $\alpha^3 = \mu\nu = \nu\mu = \nu\alpha\mu = 0$;

(11) $\alpha^3 = \mu\nu, \nu\mu = \nu\alpha^2 = \alpha^2\mu = 0$;

(12) $\alpha^4 = \mu\nu, \nu\alpha = \alpha^2\mu = 0$;

(13) $\alpha^m = \nu\alpha = \alpha\mu = (\mu\nu)^n = 0,\ 2 \leqslant m \in \mathbb{N}, 1 \leqslant n \in \mathbb{N}$;

(14) $\alpha^2 = \mu\nu, \nu\alpha\mu = 0$;

(15) $\alpha^3 = \mu\nu, \nu\alpha = \alpha^2\mu = 0$;

(16) $\alpha^3 = \mu\nu, \nu\alpha = \nu\mu = 0$;

$$\begin{tikzcd} \circ \arrow[loop left, "\alpha"] \arrow[r, shift left, "\mu"] & \circ \arrow[l, shift left, "\nu"] \arrow[loop right, "\beta"] \end{tikzcd}$$

(17) $\alpha^2 = \beta^2 = \nu\mu = \mu\nu = 0$;

(18) $\alpha^2 = \beta^m = \nu\mu = \mu\beta = \beta\nu = (\nu\alpha\mu)^n = 0,\ 2 \leqslant m \in \mathbb{N}, 1 \leqslant n \in \mathbb{N}$;

(19) $\alpha^m = \beta^n = (\nu\mu)^r = \alpha\mu = \nu\alpha = \mu\beta = \beta\nu = 0,\ 2 \leqslant m, n \in \mathbb{N}, 1 \leqslant r \in \mathbb{N}$;

(20) $\alpha^2 = \mu\nu, \beta^2 = \nu\mu, \beta\nu = 0, \alpha\mu = k\mu\beta,\ k \in K/\{0\}$;

(21) $\alpha^m = \beta^n = 0, \beta^2 = \nu\mu, \nu\alpha = \beta\nu,\ k_1\alpha^2 = \mu\nu, \alpha\mu = k_2\mu\beta$,
$k_1, k_2 \in K/\{0\}, m, n \geqslant 2, m, n \in \mathbb{N}$.

Table W

$$\begin{tikzcd} \circ \arrow[r, shift left=2, "\mu_1{,}\mu_2{,}\mu_3"] \arrow[r] \arrow[r, shift right=2] & \circ \end{tikzcd}$$

(1) $KQ(0,3,0,0)$;

$$\begin{tikzcd} \circ \arrow[r, shift left=2, "\mu_1{,}\mu_2"] \arrow[r] & \circ \arrow[l, shift left=2, "\nu"] \end{tikzcd}$$

(2) $\mu_1\nu = \mu_2\nu = 0$;

$$\begin{tikzcd} \circ \arrow[r, shift left=3, "\mu_1{,}\mu_2"] \arrow[r, shift left=1] & \circ \arrow[l, shift left=1] \arrow[l, shift left=3, "\nu_1{,}\nu_2"] \end{tikzcd}$$

(3) $\nu_2\mu_1 = \nu_1\mu_2, \mu_1\nu_1 = \mu_2\nu_1 = \mu_1\nu_2 = \mu_2\nu_2 = \nu_1\mu_1 = 0$;

$$\begin{tikzcd} \circ \arrow[loop above, "\alpha_1"] \arrow[loop below, "\alpha_2"'] \arrow[r, "\mu"] & \circ \end{tikzcd}$$

(4) $\alpha_1^2 = \alpha_2^2 = \alpha_1\alpha_2 = \alpha_2\alpha_1 = 0$,
$\alpha_1\mu = \alpha_2\mu = 0$;

$$\begin{tikzcd} \circ \arrow[loop above, "\alpha"] \arrow[r, shift left=1, "\mu_1"] \arrow[r, shift right=1, "\mu_2"'] & \circ \end{tikzcd}$$

(5) $\alpha^2 = \alpha\mu_1 = \alpha\mu_2 = 0$;

$$\begin{tikzcd} \circ \arrow[loop left, "\alpha"] \arrow[r, "\mu"] & \circ \end{tikzcd}$$

(6) $\alpha^7 = \alpha^2\mu = 0$;

(7) $\alpha^4 = \alpha^3\mu = 0$;

$$\begin{tikzcd} \circ \arrow[loop left, "\alpha"] \arrow[r, "\mu"] & \circ \arrow[loop right, "\beta"] \end{tikzcd}$$

(8) $\alpha^2 = \beta^3 = \alpha\mu = 0$;

(9) $\alpha^3 = \beta^3 = \alpha\mu = \mu\beta^2 = 0$;

(10) $\alpha^2 = \beta^4 = \alpha\mu = \mu\beta^2 = 0$;

(11) $\alpha^2 = \beta^3 = \alpha\mu\beta = \mu\beta^2 = 0$;

(12) $\alpha^4 = \beta^5 = \mu\beta^2 = 0, \alpha\mu = \mu\beta$;

(13) $\alpha^3 = \beta^7 = \mu\beta^2 = 0, \alpha\mu = \mu\beta$;

$$\begin{tikzcd} \circ \arrow[loop left, "\alpha"] \arrow[r, shift left=1, "\mu"] & \circ \arrow[l, shift left=1, "\nu"] \end{tikzcd}$$

(14) $\alpha^3 = \mu\nu = \nu\mu = \alpha^2\mu = 0$;

(15) $\alpha^3 = \mu\nu = \alpha\mu = 0$;

(16) $\alpha^3 = \mu\nu = \nu\alpha\mu = \nu\alpha^2 = \alpha^2\mu = 0$;

(17) $\alpha^4 = \mu\nu = \nu\mu = \alpha\mu = \nu\alpha^3 = 0$;

(18) $\alpha^4 = \mu\nu = \nu\mu = \nu\alpha\mu = \nu\alpha^2 = \alpha^2\mu = 0$;

(19) $\alpha^5 = \mu\nu = \nu\mu = \nu\alpha = \alpha^2\mu = 0$;

(20) $\alpha^2 = \nu\alpha = \nu\mu\nu = \alpha\mu\nu = 0$;

(21) $\alpha^2 = \nu\alpha = \mu\nu\mu = 0$;

(22) $\alpha^3 = \nu\mu = \nu\alpha = \alpha\mu\nu = \alpha^2\mu = 0$;

(23) $\alpha^2 = \mu\nu, \alpha^3 = \alpha^2\mu = 0$;

(24) $\alpha^3 = \mu\nu, \alpha^4 = \nu\mu = \nu\alpha\mu = \nu\alpha^2 = 0$;

$$\begin{tikzcd} \circ \arrow[loop left, "\alpha"] \arrow[r, shift left=1, "\mu"] & \circ \arrow[l, shift left=1, "\nu"] \arrow[loop right, "\beta"] \end{tikzcd}$$

(25) $\alpha^3 = \beta^2 = \nu\mu = \mu\nu = \nu\alpha = \mu\beta = \beta\nu = \alpha^2\mu = 0$;

(26) $\alpha^2 = \beta^2 = \nu\mu = \alpha\mu = \nu\alpha = \beta\nu = 0$;

(27) $\alpha^2 = \mu\nu, \beta^2 = \nu\mu = \alpha\mu = \mu\beta = \beta\nu\alpha = 0$;

(28) $\alpha^2 = \mu\nu, \beta^2 = \nu\mu = \alpha\mu = \beta\nu = 0$;

(29) $\alpha^2 = \mu\nu, \beta^2 = \nu\mu = \nu\alpha = \mu\beta = 0$;

(30) $\alpha^2 = \mu\nu, \beta^2 = \nu\mu = \nu\alpha = \beta\nu = \alpha\mu\beta = 0$;

(31) $\alpha\mu = \mu\beta, \alpha^2 = \beta^3 = \mu\nu = \nu\alpha = \beta\nu = \mu\beta^2 = 0$;

(32) $\alpha\mu = \mu\beta, \alpha^2 = \beta^2 = \nu\alpha = \beta\nu = \mu\nu\mu = \nu\mu\nu = 0$;

(33) $\alpha\mu = \mu\beta, \alpha^3 = \beta^3 = \nu\mu = \mu\nu = \nu\alpha = \beta\nu = \mu\beta^2 = \alpha^2\mu = 0$;

(34) $\alpha\mu = \mu\beta, \alpha^3 = \beta^2 = \nu\mu = \nu\alpha = \beta\nu = \alpha^2\mu = 0$.

A.2 Supporting materials of Example 2.3.9

We denote by $Q_s(A)$ the set of pairwise non-isomorphic basic support τ-tilting A-modules with support-rank s. In order to give a proof of Example 2.3.9, we need the following observation. Let $\mathbf{A}_n$ be the path algebra presented by

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n\ .$$

Then, the indecomposable projective $\mathbf{A}_n$-module P_1 at vertex 1 is the unique indecomposable projective-injective $\mathbf{A}_n$-module.

Lemma A.2.1 (see also [Ad2])**.** *With the above notations, any τ-tilting $\mathbf{A}_n$-module T contains P_1 as an indecomposable direct summand. Moreover, there exists a bijection*

$$Q_n(\mathbf{A}_n) \longleftrightarrow Q_{n-1}(\mathbf{A}_n)$$

given by $Q_n(\mathbf{A}_n) \ni T \longmapsto T/P_1 \in Q_{n-1}(\mathbf{A}_n)$.

Proof. Let P_i be the indecomposable projective $\mathbf{A}_n$-module at vertex i. By the poset structure on $\mathsf{s}\tau\text{-}\mathsf{tilt}\ \mathbf{A}_n$, any τ-tilting $\mathbf{A}_n$-module $M \not\simeq \mathbf{A}_n$ satisfies $M \le \mu_{P_i}^-(\mathbf{A}_n)$ for some $1 \leqslant i \leqslant n$. Since $\mu_{P_1}^-(\mathbf{A}_n)$ is support-rank $n-1$ and all $\mu_{P_i}^-(\mathbf{A}_n)$ with $i \geqslant 2$ have P_1 as a direct summand, if a τ-tilting $\mathbf{A}_n$-module M does not have P_1 as a direct summand, there must exist a τ-tilting $\mathbf{A}_n$-module $T = P_1 \oplus U$ such that $M \le \mu_{P_1}^-(T)$. Then, we look at $T = P_1 \oplus U$. If $Ue_1 = 0$, we have $\mu_{P_1}^-(T) = U$ by Definition-Theorem 2.1.4 so that $T/P_1 \in Q_{n-1}(\mathbf{A}_n)$. If $Ue_1 \neq 0$, each indecomposable direct summand V of U satisfying $Ve_1 \neq 0$ must be uniserial and $\mathsf{top}(V) = S_1$ (i.e., V is a quotient module of P_1), so that we also have $\mu_{P_1}^-(T) = U$ by Definition-Theorem 2.1.4, and hence $T/P_1 \in Q_{n-1}(\mathbf{A}_n)$. Thus, we conclude that any τ-tilting $\mathbf{A}_n$-module contains P_1 as a direct summand.

Moreover, the map $\mathsf{q} : Q_n(\mathbf{A}_n) \to Q_{n-1}(\mathbf{A}_n)$ defined by $\mathsf{q}(T) = T/P_1$ is injective since T must have P_1 as a direct summand. We show that q is also surjective. For any $U \in Q_{n-1}(\mathbf{A}_n)$, it is not difficult to check that $(\tau U)e_1 = 0$ and $\mathsf{Hom}_{\mathbf{A}_n}(P_1, \tau U) = 0$. Then, we can find $P_1 \oplus U \in Q_n(\mathbf{A}_n)$ satisfying $\mathsf{q}(P_1 \oplus U) = U$ for any $U \in Q_{n-1}(\mathbf{A}_n)$. Thus, we conclude that q is a bijection. $\square$

It is immediate that $a_n(\mathbf{A}_n) = a_{n-1}(\mathbf{A}_n)$. This can also be verified by the formula

$$a_s(\mathbf{A}_n) = \tfrac{n-s+1}{n+1}\tbinom{n+s}{s},$$

which is given by [ONFR].

In the following, we divide the proof of Example 2.3.9 into several propositions. As a beginning, we find the number $\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ \Lambda_4$ by computing the left mutations starting with Λ_4. Since the number is small, we can do this by hand.

Example A.2.2. Let P_i be the indecomposable projective Λ_4-module at vertex i. Then,

$$P_1 = \begin{smallmatrix} & 1 & \\ 2 & & 3 \\ & 4 & \end{smallmatrix}, P_2 = \begin{smallmatrix} 2 \\ 4 \end{smallmatrix}, P_3 = \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \text{ and } P_4 = 4.$$

By direct calculation, we find that (1) all τ-tilting Λ_4-modules are

$\begin{smallmatrix}&1&\\2&&3\\&4&\end{smallmatrix}\oplus\begin{smallmatrix}2\\4\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 4$	$\begin{smallmatrix}&1&\\2&&3\\&4&\end{smallmatrix}\oplus\begin{smallmatrix}2\\4\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus\begin{smallmatrix}2&&3\\&4&\end{smallmatrix}$	$\begin{smallmatrix}&1&\\2&&3\\&4&\end{smallmatrix}\oplus\begin{smallmatrix}1\\3\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus\begin{smallmatrix}&1&\\2&&3\end{smallmatrix}$	$\begin{smallmatrix}&1&\\2&&3\\&4&\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus\begin{smallmatrix}2\\4\end{smallmatrix}\oplus 2$	$\begin{smallmatrix}&1&\\2&&3\\&4&\end{smallmatrix}\oplus\begin{smallmatrix}1\\3\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 4$
$\begin{smallmatrix}&1&\\2&&3\\&4&\end{smallmatrix}\oplus\begin{smallmatrix}2\\4\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 4$	$\begin{smallmatrix}&1&\\2&&3\\&4&\end{smallmatrix}\oplus\begin{smallmatrix}2\\4\end{smallmatrix}\oplus 2\oplus\begin{smallmatrix}2&&3\\&4&\end{smallmatrix}$	$\begin{smallmatrix}&1&\\2&&3\\&4&\end{smallmatrix}\oplus\begin{smallmatrix}1\\3\end{smallmatrix}\oplus 3\oplus\begin{smallmatrix}&1&\\2&&3\end{smallmatrix}$	$\begin{smallmatrix}&1&\\2&&3\\&4&\end{smallmatrix}\oplus\begin{smallmatrix}1\\3\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 3$	$\begin{smallmatrix}&1&\\2&&3\\&4&\end{smallmatrix}\oplus 3\oplus 2\oplus\begin{smallmatrix}2&&3\\&4&\end{smallmatrix}$;
$\begin{smallmatrix}&1&\\2&&3\\&4&\end{smallmatrix}\oplus\begin{smallmatrix}1\\3\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 4$	$\begin{smallmatrix}&1&\\2&&3\\&4&\end{smallmatrix}\oplus 3\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus\begin{smallmatrix}2&&3\\&4&\end{smallmatrix}$	$\begin{smallmatrix}&1&\\2&&3\\&4&\end{smallmatrix}\oplus 2\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus\begin{smallmatrix}&1&\\2&&3\end{smallmatrix}$	$1\oplus\begin{smallmatrix}1\\3\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 4$	$\begin{smallmatrix}&1&\\2&&3\\&4&\end{smallmatrix}\oplus 2\oplus 3\oplus\begin{smallmatrix}&1&\\2&&3\end{smallmatrix}$

(2) all support τ-tilting Λ_4-modules with support-rank 3 are

$\begin{smallmatrix}1\\3\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus\begin{smallmatrix}&1&\\2&&3\end{smallmatrix}$	$2\oplus 3\oplus\begin{smallmatrix}&1&\\2&&3\end{smallmatrix}$	$3\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus\begin{smallmatrix}2&&3\\&4&\end{smallmatrix}$	$\begin{smallmatrix}1\\3\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 4$	$\begin{smallmatrix}2\\4\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 4$	$1\oplus\begin{smallmatrix}1\\3\end{smallmatrix}\oplus 4$
$\begin{smallmatrix}1\\3\end{smallmatrix}\oplus 3\oplus\begin{smallmatrix}&1&\\2&&3\end{smallmatrix}$	$\begin{smallmatrix}2\\4\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus\begin{smallmatrix}2&&3\\&4&\end{smallmatrix}$	$3\oplus 2\oplus\begin{smallmatrix}2&&3\\&4&\end{smallmatrix}$	$\begin{smallmatrix}2\\4\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 4$	$\begin{smallmatrix}2\\4\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 2$	$1\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 4$;
$2\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus\begin{smallmatrix}&1&\\2&&3\end{smallmatrix}$	$\begin{smallmatrix}2\\4\end{smallmatrix}\oplus 2\oplus\begin{smallmatrix}2&&3\\&4&\end{smallmatrix}$	$1\oplus\begin{smallmatrix}1\\3\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\end{smallmatrix}$	$\begin{smallmatrix}1\\3\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 3$		

(3) all support τ-tilting Λ_4-modules with support-rank 2 are

$\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 4$	$\begin{smallmatrix}2\\4\end{smallmatrix}\oplus 4$	$\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 2$	$\begin{smallmatrix}1\\3\end{smallmatrix}\oplus 3$	$1\oplus 4$	$\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 3$	$\begin{smallmatrix}2\\4\end{smallmatrix}\oplus 2$	$\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 1$	$\begin{smallmatrix}1\\3\end{smallmatrix}\oplus 1$	$2\oplus 3$.

Combining the above with $a_0(\Lambda_4)=1$ and $a_1(\Lambda_4)=4$, we conclude that $\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ \Lambda_4=46$.

Proposition A.2.3. *For any $n\geqslant 4$ and $1\leqslant s\leqslant n-3$, we have*

$$a_s(\Lambda_n)=a_s(\Lambda_{n-1})+a_{s-1}(\Lambda_n).$$

Proof. Let e_i be the idempotent of Λ_n at vertex i. For any support τ-tilting Λ_n-module M satisfying $Me_n=0$, it is obvious that M is a support τ-tilting Λ_{n-1}-module. Then, let $Q_s(\Lambda_n;e_n)$ be the set of the support τ-tilting Λ_n-modules T with support-rank s and $Te_n\neq 0$. We show that there is a bijection

$$\mathfrak{q}:Q_s(\Lambda_n;e_n)\longrightarrow Q_{s-1}(\Lambda_n),$$

and then, the statement follows from this bijection.

Let X be an indecomposable Λ_n-module with support-rank $t\leqslant n-3$ and $Xe_n\neq 0$. Then, X is an indecomposable module over a path algebra of type $\mathbb{A}$ and it corresponds to a root so that X is of the form

$$\begin{smallmatrix}S_{n-t+1}\\ \vdots\\ S_{n-1}\\ S_n\end{smallmatrix}.$$

In this case, we denote X by $[n-t+1,n]$.

Let $T\in Q_s(\Lambda_n;e_n)$ and (T,P) the corresponding support τ-tilting pair. There exists at least one indecomposable direct summand of T, say X, which satisfies $Xe_n\neq 0$ and we choose $X:=[n-t+1,n]$ of the largest possible length t.

(1) We show that $Te_m = 0$ for any arrow $m \longrightarrow n-t+1$. In fact, if $t = s$, it is true since T is support-rank s. If $t \leqslant s-1$, the inequality $n-t+1 \geqslant 5$ makes m unique and $m = n-t$. By the maximality of X, the number of indecomposable direct summands X' of T with $X'e_n \neq 0$ is at most t and $t + |P| = t + n - s \leqslant n-1$. (Note that $X'e_{n-t} = 0$ is obvious.) We consider the remaining indecomposable direct summand Y of T satisfying $Ye_n = 0$. Suppose that $Ye_{n-t} \neq 0$. It is enough to consider the following five types of Y:

$$\begin{array}{ccc} & S_1 & \\ S_2 & & S_3 \\ & S_4 & \\ & \vdots & \\ & S_{n-t} & \\ & \vdots & \\ & S_a & \end{array}, \quad \begin{array}{ccc} S_2 & & S_3 \\ & S_4 & \\ & \vdots & \\ & S_{n-t} & \\ & \vdots & \\ & S_a & \end{array}, \quad \begin{array}{c} S_2 \\ S_4 \\ \vdots \\ S_{n-t} \\ \vdots \\ S_a \end{array}, \quad \begin{array}{c} S_3 \\ S_4 \\ \vdots \\ S_{n-t} \\ \vdots \\ S_a \end{array}, \quad \begin{array}{c} S_4 \\ \vdots \\ S_{n-t} \\ \vdots \\ S_a \end{array},$$

where $4 \leqslant n-t \leqslant a \leqslant n-1$. One can check that $[n-t+1, a+1]$ is a submodule of τY for any type above. Then, $\mathsf{Hom}_{\Lambda_n}(X, \tau Y) \neq 0$ and it contradicts with $T \in Q_s(\Lambda_n; e_n)$. Therefore, we must have $Ye_{n-t} = 0$.

(2) According to (1), we can divide T into a direct sum $W \oplus Z$ such that the support of W is $\{e_{n-t+1}, \ldots, e_{n-1}, e_n\}$, the supports of W and Z are disjoint and the support of Z does not contain e_m with $m \longrightarrow n-t+1$. Then, we define

$$\Lambda_{[n-t+1,n]} := \Lambda_n / < e_1 + e_2 + \cdots + e_{n-t} >.$$

Since $T \in Q_s(\Lambda_n; e_n)$ and the supports of W and Z are disjoint, W is actually a support τ-tilting Λ_n-module by repeatedly calculating the left mutations started at direct summands of Z. By Proposition 2.3.8, $|W| = t$ since the support-rank of W is t. Then, W becomes a τ-tilting $\Lambda_{[n-t+1,n]}$-module. We note that X is the unique indecomposable projective-injective $\Lambda_{[n-t+1,n]}$-module since $\Lambda_{[n-t+1,n]}$ is isomorphic to the path algebra $\mathbf{A}_t$. By Lemma A.2.1, the quotient module W/X is a support τ-tilting $\Lambda_{[n-t+1,n]}$-module with support-rank $t-1$.

(3) Based on the analysis in (1) and (2), we define $U := T/X$ for any $T \in Q_s(\Lambda_n; e_n)$. Since T is a support τ-tilting Λ_n-module with support-rank s, U is a support τ-tilting Λ_n-module with support-rank $s-1$ by $W/X \in Q_{t-1}(\Lambda_{[n-t+1,n]})$ and the fact that the supports of W and Z are disjoint. Thus, we may define the map from $Q_s(\Lambda_n; e_n)$ to $Q_{s-1}(\Lambda_n)$ by $\mathfrak{q}(T) = U$.

Next, we show that the map $\mathfrak{q}$ defined above is a bijection. On the one hand, we know that $\mathfrak{q}$ is an injection. By the analysis in (2), we may define $T_1 := Z_1 \oplus X_1 \oplus V_1 \in Q_s(\Lambda_n; e_n)$ and $T_2 := Z_2 \oplus X_2 \oplus V_2 \in Q_s(\Lambda_n; e_n)$ such that

- $X_i = [n - t_i + 1, n]$ for $i = 1, 2$,
- the support of Z_1 is included in $\{e_1, e_2, \ldots, e_{n-t_1-1}\}$,
- the support of Z_2 is included in $\{e_1, e_2, \ldots, e_{n-t_2-1}\}$,

- the support of V_1 is $\{e_{n-t_1+1}, e_{n-t_1+2}, \ldots, e_{i_1-1}, e_{i_1+1}, \ldots, e_n\}$ with exactly one e_{i_1} satisfying $V_1 e_{i_1} = 0$ for $n - t_1 + 1 \leqslant i_1 \leqslant n$,
- the support of V_2 is $\{e_{n-t_2+1}, e_{n-t_2+2}, \ldots, e_{i_2-1}, e_{i_2+1}, \ldots, e_n\}$ with exactly one e_{i_2} satisfying $V_2 e_{i_2} = 0$ for $n - t_2 + 1 \leqslant i_2 \leqslant n$.

Obviously, $X_1 \neq X_2$ if and only if $t_1 \neq t_2$. If $X_1 = X_2$, $T_1 \neq T_2$ implies $Z_1 \oplus V_1 \neq Z_2 \oplus V_2$. Then, we assume that $X_1 \neq X_2$. If we list the idempotents by increasing the subscripts, the last two idempotents outside of the support of $Z_1 \oplus V_1$ must be $\{e_{n-t_1}, e_{i_1}\}$ and the last two idempotents outside of the support of $Z_2 \oplus V_2$ must be $\{e_{n-t_2}, e_{i_2}\}$. Since $t_1 \neq t_2$, we have $e_{n-t_1} \neq e_{n-t_2}$ so that the supports of $Z_1 \oplus V_1$ and $Z_2 \oplus V_2$ are different. Thus, we conclude that $\mathfrak{q}(T_1) \neq \mathfrak{q}(T_2)$ if $T_1 \neq T_2 \in Q_s(\Lambda_n; e_n)$.

On the other hand, $\mathfrak{q}$ is a surjection. We assume that $U \in Q_{s-1}(\Lambda_n)$. Since $s-1 \leqslant n-4$, there are at least 4 idempotents of Λ_n outside of the support of U.

- If there are exactly 4 idempotents e_1, e_2, e_3 and e_i with $4 \leqslant i \leqslant n$ outside of the support of U, then $s = n - 3$ and U becomes a support τ-tilting $\Lambda_{[4,n]}$-module with support-rank $n - 4$. Let $P := [4, n]$ be the indecomposable projective $\Lambda_{[4,n]}$-module at vertex 4. Since $\Lambda_{[4,n]}$ is isomorphic to the path algebra $\mathbf{A}_{n-3}$, we have $T := P \oplus U \in Q_{n-3}(\Lambda_n; e_n)$ by Lemma A.2.1, and T maps to U.
- Otherwise, there are at least two idempotents in $\{e_4, e_5, \ldots, e_n\}$ outside of the support of U. Let $j > i > \cdots \geqslant 4$ be the first two numbers in decreasing order of such subscripts. Then, e_{i+1} does not appear in the support of τU, and we can find an indecomposable projective Λ_n-module $P := [i+1, n]$ such that $\mathsf{Hom}_{\Lambda_n}(P, \tau U) = 0$. Hence, $T := P \oplus U \in Q_s(\Lambda_n; e_n)$. In fact, $Te_j \neq 0$ and $Te_{j'} = 0$ for any $j' \neq j$ satisfying $Ue_{j'} = 0$. Then, T maps to U.

Therefore, $\mathfrak{q}$ is a surjection. $\square$

We denote by $[2, 4, \ldots, n]$ (resp., $[3, 4, \ldots, n]$) the indecomposable projective Λ_n-module at vertex 2 (resp., 3).

Proposition A.2.4. *For any $n \geqslant 4$, we have*

$$a_{n-2}(\Lambda_n) = a_{n-2}(\Lambda_{n-1}) + a_{n-3}(\Lambda_n) + a_{n-3}(\mathbf{A}_{n-3}).$$

Proof. Let e_i be the idempotent of Λ_n at vertex i. For any support τ-tilting Λ_n-module M satisfying $Me_n = 0$, it is obvious that M is a support τ-tilting Λ_{n-1}-module. Then, the number of such modules with support-rank $n-2$ is $a_{n-2}(\Lambda_{n-1})$.

Let $Q_{n-2}(\Lambda_n; e_n)$ be the set of support τ-tilting Λ_n-modules T with support-rank $n-2$ and $Te_n \neq 0$. For any $T \in Q_{n-2}(\Lambda_n; e_n)$, we denote it by $T = X \oplus U$ with an indecomposable direct summand X satisfying $Xe_n \neq 0$. We may set $X := [n-t+1, n]$ of the largest possible length t, while $X = [2, 4, \ldots, n]$ is also allowed if $t = n-2$. We show that $\mu_X^-(T) = U$ and therefore, $U \in Q_{n-3}(\Lambda_n)$. Then, we can define a map $\mathfrak{q}$ from $Q_{n-2}(\Lambda_n; e_n)$ to $Q_{n-3}(\Lambda_n)$ by $\mathfrak{q}(T) = U$.

- If $t \leqslant n-3$, this is similar to the situation in the proof of Proposition A.2.3. Thus, $Te_m = 0$ for any arrow $m \longrightarrow n-t+1$ such that $T = X \oplus V \oplus Z$, where $X \oplus V$ is a τ-tilting $\Lambda_{[n-t+1,n]}$-module with the unique indecomposable projective-injective $\Lambda_{[n-t+1,n]}$-module X, the supports of $X \oplus V$ and Z are disjoint and the support of Z does not contain e_m with $m \longrightarrow n-t+1$. By Lemma A.2.1, we have $\mu_X^-(T) = V \oplus Z$. In this case, $\mathfrak{q}(T_1) \neq \mathfrak{q}(T_2)$ if $T_1 \neq T_2 \in Q_{n-2}(\Lambda_n; e_n)$.
- Let $t = n-2$, the support of X is either $\{e_2, e_4, \ldots, e_n\}$ or $\{e_3, e_4, \ldots, e_n\}$ such that X is uniquely determined (since the support of X cannot contain all the idempotents $\{e_2, e_3, e_4, \ldots, e_n\}$). Then, $Te_1 = 0$ is obvious and $\mu_X^-(T) = U$ is also true. In fact, let $T := X \oplus U$ and $X := [2, 4, \ldots, n]$. The condition $Te_1 = Te_3 = 0$ makes T to be a τ-tilting $\mathbf{A}_{n-2}$-module and makes X to be the unique indecomposable projective-injective $\mathbf{A}_{n-2}$-module. By Lemma A.2.1, we deduce that $\mu_X^-(T) = U$. Similarly, one can observe the fact $\mu_X^-(T) = U$ for $X = [3, 4, \ldots, n]$.

Let $X_1 := [2, 4, \ldots, n]$ and $X_2 := [3, 4, \ldots, n]$. We observe that the map $\mathfrak{q}$ defined above is not an injection because $\mathfrak{q}(X_1 \oplus U) = \mathfrak{q}(X_2 \oplus U) = U$ whenever U is a τ-tilting $\Lambda_{[4,n]}$-module, which appears in the case $t = n-2$. Also, it will be useful to mention that for any $T \in Q_{n-2}(\Lambda_n; e_n)$, if there exists an idempotent $e_i \in \{e_4, e_5, \ldots, e_n\}$ satisfying $Te_i = 0$, then $t \leqslant n-i \leqslant n-4$.

Next, we show that the map $\mathfrak{q} : Q_{n-2}(\Lambda_n; e_n) \to Q_{n-3}(\Lambda_n)$ is a surjection. For any $U \in Q_{n-3}(\Lambda_n)$, there exist exactly three idempotents of Λ_n outside of the support of U.

- Suppose that there are at least two idempotents in $\{e_4, e_5, \ldots, e_n\}$ outside of the support of U. Let $j > i \geqslant 4$ (or $j > i > k \geqslant 4$ if k exists) be the order of such numbers. Then, e_{i+1} does not appear in the support of τU, so that $\mathsf{Hom}_{\Lambda_n}(P, \tau U) = 0$ with the indecomposable projective Λ_n-module $P := [i+1, n]$. Hence, $P \oplus U \in Q_{n-2}(\Lambda_n; e_n)$ maps to U.

  Note that for any Y satisfying $Y \oplus U \in Q_{n-2}(\Lambda_n; e_n)$ and $\mathfrak{q}(Y \oplus U) = U$, the support-rank of $Y \oplus U$ is the support-rank of U plus 1. Then, there always exists an idempotent $e_k \in \{e_4, e_5, \ldots, e_n\}$ satisfying $(Y \oplus U)e_k = 0$, so that $Y \oplus U$ is included in the case $t \leqslant n-4$. By the injectivity of $\mathfrak{q}$ in the case $t \leqslant n-3$, we conclude that $P \oplus U \in Q_{n-2}(\Lambda_n; e_n)$ is uniquely determined in this case.
- Suppose that there is exactly one idempotent e_i in $\{e_4, e_5, \ldots, e_n\}$ outside of the support of U. This is equivalent to saying that there are exactly two idempotents of e_1, e_2, e_3 outside of the support of U.

  If $Ue_1 \neq 0, Ue_2 = 0, Ue_3 = 0$, then $U = S_1 \oplus V$, where V is a support τ-tilting $\Lambda_{[4,n]}$-module with support-rank $n-4$. We observe that $(U, P_2 \oplus P_3)$ is an almost complete support τ-tilting pair, so that it has two completions and one of which is $(U, P_2 \oplus P_3 \oplus P_i)$. Since $Ue_j \neq 0$ for any $i \neq j \geqslant 4$, $(U, P_2 \oplus P_3 \oplus P_j)$ cannot be a support τ-tilting pair and then, the other completion must be of the form

$(U \oplus Y, P_2 \oplus P_3)$. In particular, $Ye_2 = Ye_3 = 0$ holds. Since Y is indecomposable and cannot be S_1, $Ye_1 = 0$ also holds. Note that the support-rank of $Y \oplus U$ is the support-rank of U plus 1, we must have $Ye_i \neq 0$. Then, $Y \oplus V$ is a τ-tilting $\Lambda_{[4,n]}$-module. Since V is support-rank $n-4$, it cannot have $[4,n]$ as a direct summand. By Lemma A.2.1, $Y = [4,n]$ is unique and $Y \oplus U \in Q_{n-2}(\Lambda_n; e_n)$. Moreover, $\mathfrak{q}(Y \oplus U) = U$.

If $Ue_1 = 0, Ue_2 \neq 0, Ue_3 = 0$, similar to the above case, we can find a Y satisfying $Ye_1 = Ye_3 = 0$ such that $Y \oplus U \in Q_{n-2}(\Lambda_n; e_n)$ maps to U. Then, U can be considered as a support τ-tilting $\mathbf{A}_{n-2}$-module with support-rank $n-3$ as well as $Y \oplus U$ can be considered as a τ-tilting $\mathbf{A}_{n-2}$-module. By Lemma A.2.1, Y is unique and it must be X_1.

If $Ue_1 = 0, Ue_2 = 0, Ue_3 \neq 0$, similar to above, $X_2 \oplus U \in Q_{n-2}(\Lambda_n; e_n)$ maps to U and X_2 is uniquely determined by Lemma A.2.1.

- Suppose that e_1, e_2, e_3 are outside of the support of U. Then, U is actually a τ-tilting $\Lambda_{[4,n]}$-module and $[4,n]$ must be an indecomposable (projective) direct summand of U following Lemma A.2.1. Then, $\mathsf{Hom}_{\Lambda_n}(X_i, \tau U) = 0$ for $i = 1, 2$ implies that $X_1 \oplus U, X_2 \oplus U \in Q_{n-2}(\Lambda_n; e_n)$ and both of them are mapped to U.

  We show that X_1 and X_2 are the only possible cases. Assume that $Y \oplus U \in Q_{n-2}(\Lambda_n; e_n)$ maps to U. Then, only one of Ye_1, Ye_2, Ye_3 is not zero. If $Ye_1 \neq 0$, Y must be S_1 since Y is indecomposable and $Ye_2 = Ye_3 = 0$. However, this contradicts with the fact $Ye_n \neq 0$ deduced by $\mathfrak{q}(Y \oplus U) = U$. Thus, we have either $Ye_1 = Ye_2 = 0$, $Ye_3 \neq 0$, $Ye_n \neq 0$ or $Ye_1 = Ye_3 = 0$, $Ye_2 \neq 0$, $Ye_n \neq 0$, so that $Y = X_1$ or X_2.

Now, we found that the map $\mathfrak{q} : Q_{n-2}(\Lambda_n; e_n) \to Q_{n-3}(\Lambda_n)$ is indeed a surjection. If one wants to use this surjection to count the number of modules in $Q_{n-2}(\Lambda_n; e_n)$, one should note that both $X_1 \oplus U$ and $X_2 \oplus U$ are mapped to U whenever U is a τ-tilting $\Lambda_{[4,n]}$-module, and it is the only part that needs to be double calculated. These are exactly the $a_{n-3}(\Lambda_{[4,n]}) = a_{n-3}(\mathbf{A}_{n-3})$ pairs of modules in $Q_{n-2}(\Lambda_n; e_n)$. Therefore, the number of modules in $Q_{n-2}(\Lambda_n; e_n)$ is $a_{n-3}(\Lambda_n) + a_{n-3}(\mathbf{A}_{n-3})$. □

We define $\mathbb{A}_2^1 := \mathbf{A}_2$ and $\mathbb{A}_n^1 := KQ/ < \alpha\beta >$ for any $n \geqslant 3$, where

$$Q: \ 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n\,.$$

Proposition A.2.5. *Let $n \geqslant 4$, we have*

(1) $a_s(\mathbb{A}_n^1) = a_s(\mathbb{A}_{n-1}^1) + a_{s-1}(\mathbb{A}_n^1)$ *for any* $1 \leqslant s \leqslant n-2$.

(2) $a_{n-1}(\mathbb{A}_n^1) = a_{n-1}(\mathbb{A}_{n-1}^1) + a_{n-1}(\mathbf{A}_{n-1}) + a_{n-2}(\mathbf{A}_{n-2}) + \sum\limits_{i=3}^{n-1} a_{i-1}(\mathbb{A}_{i-1}^1) \cdot a_{n-i}(\mathbf{A}_{n-i})$.

(3) $a_n(\mathbb{A}_n^1) = a_{n-1}(\mathbf{A}_{n-1}) + a_{n-2}(\mathbf{A}_{n-2})$.

Proof. (1) The proof is similar to the proof of Proposition A.2.3. We omit the details.

(2) Let $T \in Q_{n-1}(\mathbb{A}_n^1)$. There exists exactly one idempotent e_i such that $Te_i = 0$. If $i = 1$, T becomes a τ-tilting $\mathbf{A}_{n-1}$-module. If $i = 2$, we can divide T into a direct sum $T_1 \oplus T_2$ such that T_1 is the unique τ-tilting $\mathbf{A}_1$-module and T_2 is a τ-tilting $\mathbf{A}_{n-2}$-module. If $3 \leqslant i \leqslant n-1$, we can divide T into a direct sum $T_1 \oplus T_2$ such that T_1 is a τ-tilting $\mathbb{A}_{i-1}^1$-module and T_2 is a τ-tilting $\mathbf{A}_{n-i}$-module. If $i = n$, T is a τ-tilting $\mathbb{A}_{n-1}^1$-module. Hence, we get the formula.

(3) Let $T \in Q_n(\mathbb{A}_n^1)$. Similar to the proof of Lemma A.2.1, we find that T always contains $\frac{1}{2}$ as a direct summand. If $T = \frac{1}{2} \oplus U$ with $Ue_1 = 0$, U becomes a τ-tilting $\mathbf{A}_{n-1}$-module. If $T = \frac{1}{2} \oplus U$ with $Ue_1 \neq 0$, one may check that $U = 1 \oplus V$ with $Ve_1 = 0$. Since $\tau(1) = S_2$ and U is τ-rigid, we have $Ve_2 = 0$. Thus, V is a τ-tilting $\mathbf{A}_{n-2}$-module. $\square$

By combining the above proposition and $a_n(\mathbf{A}_n) = \frac{1}{n+1}\binom{2n}{n}$, we have

Corollary A.2.6. $a_n(\mathbb{A}_n^1) = \frac{1}{n}\binom{2n-2}{n-1} + \frac{1}{n-1}\binom{2n-4}{n-2}$ *is the sequence* A005807 *in [Sl].*

Let $\mathbf{D}_n$ be the path algebra presented by

$$\begin{tikzcd}[row sep=small] 1 \arrow[dr] & & & & & \\ & 3 \arrow[r] & 4 \arrow[r] & \cdots \arrow[r] & n-1 \arrow[r] & n \, . \\ 2 \arrow[ur] & & & & & \end{tikzcd}$$

We recall $a_n(\mathbf{D}_n) = \frac{3n-4}{2n-2}\binom{2n-2}{n-2}$ from [ONFR].

Proposition A.2.7. *For any* $n \geqslant 5$*, we have*

$$a_{n-1}(\Lambda_n) = a_{n-1}(\Lambda_{n-1}) + a_{n-1}(\mathbf{D}_{n-1}) + 2a_{n-1}(\mathbb{A}_{n-1}^1) + \sum_{i=4}^{n-1} a_{i-1}(\Lambda_{i-1}) \cdot a_{n-i}(\mathbf{A}_{n-i}).$$

Proof. Let $T \in Q_{n-1}(\Lambda_n)$. There is exactly one idempotent e_i such that $Te_i = 0$. If $i = 1$, T becomes a τ-tilting $\mathbf{D}_{n-1}$-module. If $i = 2$ or 3, then $\beta\nu = 0$ or $\alpha\mu = 0$, and T becomes a τ-tilting $\mathbb{A}_{n-1}^1$-module. If $4 \leqslant i \leqslant n-1$, we can divide T into a direct sum $T_1 \oplus T_2$ such that T_1 is a τ-tilting Λ_{i-1}-module and T_2 is a τ-tilting $\mathbf{A}_{n-i}$-module. If $i = n$, T is a τ-tilting Λ_{n-1}-module. Hence, we deduce the formula. $\square$

Proposition A.2.8. *For any* $n \geqslant 4$*, we have*

$$a_n(\Lambda_n) = a_{n-1}(\Lambda_n) - a_{n-3}(\mathbf{A}_{n-3}).$$

Proof. Let P_i be the indecomposable projective Λ_n-module at vertex i. We explain the relation between $Q_n(\Lambda_n)$ and $Q_{n-1}(\Lambda_n)$ as follows.

(1) We show that a τ-tilting Λ_n-module either has P_1 as a direct summand or is of form $1 \oplus \frac{1}{2} \oplus \frac{1}{3} \oplus V$ with V to be a τ-tilting $\Lambda_{[4,n]}$-module. By the poset structure on $\mathsf{s\tau\text{-}tilt}\,\Lambda_n$, any τ-tilting Λ_n-module $M \not\simeq \Lambda_n$ satisfies $M \leq \mu_{P_i}^-(\Lambda_n)$ for some $1 \leqslant i \leqslant n$. Since $\mu_{P_1}^-(\Lambda_n)$ is not τ-tilting and all $\mu_{P_i}^-(\Lambda_n)$ with $i \geqslant 2$ have P_1 as a direct summand, if M does not have P_1 as a direct summand, then there must exist a τ-tilting Λ_n-module $T := P_1 \oplus U$ such that $M \leq \mu_{P_1}^-(T)$. Immediately, we have

Case (a). If $T = P_1 \oplus U$ with $Ue_1 = 0$, then $\mu^-_{P_1}(T) = U$ by Definition-Theorem 2.1.4 and $U \in Q_{n-1}(\Lambda_n)$ by Proposition 2.3.8.

Suppose that $Ue_1 \neq 0$. We remark that U does not have S_1 as a direct summand since $\mathsf{Hom}_{\Lambda_n}(P_1, \tau S_1) \neq 0$. We define

$$M_a := \begin{smallmatrix} & 1 & \\ 2 & & 3 \\ & 4 & \\ & \vdots & \\ & a & \end{smallmatrix}$$

with $4 \leqslant a \leqslant n-1$. Then, U has at least one of $\begin{smallmatrix}1\\2\end{smallmatrix}$, $\begin{smallmatrix}1\\3\end{smallmatrix}$, $\begin{smallmatrix} & 1 & \\ 2 & & 3\end{smallmatrix}$ and M_a as a direct summand, because these modules are τ-rigid and $Ue_1 \neq 0$. In particular, it is worth mentioning that the unique non-zero morphism $f : P_1 \to X$ for any $X \in \{\begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}1\\3\end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 2 & & 3\end{smallmatrix}, M_a\}$, is actually the projective cover of X and $\mathsf{coker}\ f = 0$.

(a1) If $U = \begin{smallmatrix}1\\2\end{smallmatrix} \oplus V$ with $Ve_1 = 0$, we have $\mu^-_{P_1}(T) = U$ by substituting $\mathsf{coker}\ f = 0$ into Definition-Theorem 2.1.4 and hence, $U \in Q_{n-1}(\Lambda_n)$. Since $\tau(\begin{smallmatrix}1\\2\end{smallmatrix}) = S_3$, we have $S_3 \not\subseteq \mathsf{top}\ V$ so that $Ve_3 = 0$. This implies that $Ue_3 = 0$ and $Ue_i \neq 0$ for any $i \neq 3$.

(a2) If $U = \begin{smallmatrix}1\\3\end{smallmatrix} \oplus V$ with $Ve_1 = 0$, we also have $\mu^-_{P_1}(T) = U$ and $U \in Q_{n-1}(\Lambda_n)$. Since $\tau(\begin{smallmatrix}1\\3\end{smallmatrix}) = S_2$, we have $Ue_2 = 0$ and $Ue_i \neq 0$ for any $i \neq 2$.

(a3) If U has exactly one of $\begin{smallmatrix} & 1 & \\ 2 & & 3\end{smallmatrix}$ and M_a as a direct summand, we have $\mu^-_{P_1}(T) = U$ by substituting $\mathsf{coker}\ f = 0$ into Definition-Theorem 2.1.4 and hence, $U \in Q_{n-1}(\Lambda_n)$. This implies that $Ue_i = 0$ for exactly one i with $4 \leqslant i \leqslant n$.

(a4) If U has two of $\begin{smallmatrix}1\\2\end{smallmatrix}$, $\begin{smallmatrix}1\\3\end{smallmatrix}$, $\begin{smallmatrix} & 1 & \\ 2 & & 3\end{smallmatrix}$ and M_a as direct summands and one of the direct summands is $\begin{smallmatrix} & 1 & \\ 2 & & 3\end{smallmatrix}$ or M_a, we also have $\mu^-_{P_1}(T) = U$ since there exist epimorphisms from $\begin{smallmatrix} & 1 & \\ 2 & & 3\end{smallmatrix}$ or M_a to $\begin{smallmatrix}1\\2\end{smallmatrix}$ and $\begin{smallmatrix}1\\3\end{smallmatrix}$. Then, $U \in Q_{n-1}(\Lambda_n)$. Since we can make sure that $Ue_2 \neq 0$ and $Ue_3 \neq 0$, we have $Ue_i = 0$ for exactly one i with $4 \leqslant i \leqslant n$.

(a5) If U has more than two of $\begin{smallmatrix}1\\2\end{smallmatrix}$, $\begin{smallmatrix}1\\3\end{smallmatrix}$, $\begin{smallmatrix} & 1 & \\ 2 & & 3\end{smallmatrix}$ and M_a as direct summands, it must have $\begin{smallmatrix} & 1 & \\ 2 & & 3\end{smallmatrix}$ or M_a as a direct summand. Then, similar to the above, we have $\mu^-_{P_1}(T) = U$ and $U \in Q_{n-1}(\Lambda_n)$. This also implies that $Ue_i = 0$ for exactly one i with $4 \leqslant i \leqslant n$.

Then, it remains to consider

Case (b). If $T = P_1 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus V$ with $Ve_1 = 0$, then $Ve_2 = Ve_3 = 0$, so that V becomes a τ-tilting $\Lambda_{[4,n]}$-module. In fact, if $Ve_2 \neq 0$, it implies $S_2 \subseteq \mathsf{top}\ V$ so that $\mathsf{Hom}_{\Lambda_n}(V, S_2) \neq 0$. Then, $\tau(\begin{smallmatrix}1\\3\end{smallmatrix}) = S_2$ implies that Ve_2 must be zero. Similarly, $\tau(\begin{smallmatrix}1\\2\end{smallmatrix}) = S_3$ makes Ve_3 to be zero. Then, we have $\mu^-_{P_1}(T) = 1 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus V$ by simple observation.

Now, we can claim that if a τ-tilting Λ_n-module M does not have P_1 as a direct summand, $M \leq T' := 1 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus V$ with a τ-tilting $\Lambda_{[4,n]}$-module V. Then, we observe that the left mutations of T' with respect to $\begin{smallmatrix}1\\2\end{smallmatrix}$ and $\begin{smallmatrix}1\\3\end{smallmatrix}$ are not τ-tilting, and the left mutation of T' with respect to one of direct summands of V is equivalent to that of a τ-tilting $\Lambda_{[4,n]}$-module V. Therefore, $1 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix}$ must remain in M as a direct summand.

Finally, we conclude that if a τ-tilting Λ_n-module M does not have P_1 as a direct summand, it is of form $1 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus V$ with a τ-tilting $\Lambda_{[4,n]}$-module V. Moreover, if a τ-tilting Λ_n-module T has P_1 as a direct summand, then $T/P_1 \in Q_{n-1}(\Lambda_n)$ if and only if

$T \not\simeq P_1 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus V$ with a τ-tilting $\Lambda_{[4,n]}$-module V. (This implies that T/P_1 is not always included in $Q_{n-1}(\Lambda_n)$. This is also the reason why we distinguish the following set $\mathcal{S}$.)

(2) We may construct a map $\mathfrak{q}$ from $Q_{n-1}(\Lambda_n)$ to $Q_n(\Lambda_n) \setminus \mathcal{S}$, where

$$\mathcal{S} := \left\{ P_1 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus V \mid V \text{ is a } \tau\text{-tilting } \Lambda_{[4,n]}\text{-module} \right\}.$$

Let $U \in Q_{n-1}(\Lambda_n)$, it is obvious that U does not have P_1 as a direct summand since P_1 is sincere. We first consider the case that U has S_1 as a direct summand. Since $\tau(1) = {}_{2}{}^{1}{}_{3}$, we know that U does not have one of ${}_{2}{}^{1}{}_{3}$, M_a, the indecomposable module N_2 with $\mathsf{top}\ N_2 = S_2$ and the indecomposable module N_3 with $\mathsf{top}\ N_3 = S_3$ as a direct summand. Since $U \in Q_{n-1}(\Lambda_n)$, there exists exactly one idempotent e_i with $i \neq 1$ satisfying $Ue_i = 0$.

(i) If $i = 2$, the only possible direct summand Y of U satisfying $Ye_3 \neq 0$ is $\begin{smallmatrix}1\\3\end{smallmatrix}$ and the remaining direct summands give a τ-tilting $\Lambda_{[4,n]}$-module V, so that $U = 1 \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus V$. In this subcase, $\mathfrak{q}$ is defined by mapping U to $1 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus V$. To observe that the latter one is included in $Q_n(\Lambda_n) \setminus \mathcal{S}$, we have $\mathsf{Hom}_{\Lambda_n}(\begin{smallmatrix}1\\2\end{smallmatrix} \oplus U, \tau(\begin{smallmatrix}1\\2\end{smallmatrix} \oplus U)) = 0$ since $\tau(\begin{smallmatrix}1\\2\end{smallmatrix}) = 3$ and $\tau(\begin{smallmatrix}1\\3\end{smallmatrix}) = 2$. Moreover, it is easy to check that $\mathfrak{q}$ in this subcase is a bijection.

If $i = 3$, $U = 1 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus V$ with a τ-tilting $\Lambda_{[4,n]}$-module V. Similarly, $\mathfrak{q}$ is defined by mapping U to $1 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus V \in Q_n(\Lambda_n) \setminus \mathcal{S}$, and $\mathfrak{q}$ in this subcase is also a bijection.

Note that the number of U in the cases of $i = 2, 3$ is $2a_{n-3}(\Lambda_{[4,n]}) = 2a_{n-3}(\mathbf{A}_{n-3})$.

(ii) If $i \geqslant 4$, the conditions $Ue_2 \neq 0$ and $Ue_3 \neq 0$ must imply that $U = 1 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus Z$ with $Ze_1 = Ze_2 = Ze_3 = 0$ and $Z \in Q_{n-4}(\Lambda_{[4,n]})$. We may regard Z as a support τ-tilting $\mathbf{A}_{n-3}$-module with support-rank $n-4$. By Lemma A.2.1, Z is the left mutation of τ-tilting $\Lambda_{[4,n]}$-module $V := P \oplus Z$ with $P = [4, n]$. Then, $1 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus V \in Q_n(\Lambda_n) \setminus \mathcal{S}$ is obvious. In this subcase, $\mathfrak{q}$ is defined by mapping U to $1 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus V$ and it is a bijection. Besides, the number of U in this subcase is $a_{n-4}(\Lambda_{[4,n]}) = a_{n-4}(\mathbf{A}_{n-3})$.

Similar to the situation in Proposition A.2.4, the map $\mathfrak{q}$ defined in the case where U has S_1 as a direct summand is no longer a bijection, but a surjection. If we count the number of modules in $Q_n(\Lambda_n) \setminus \mathcal{S}$ in this case, there should be $a_{n-3}(\mathbf{A}_{n-3}) + a_{n-4}(\mathbf{A}_{n-3})$ overlaps.

Next, we consider $U \in Q_{n-1}(\Lambda_n)$ such that $Ue_1 \neq 0$ and U does not have S_1 as a direct summand. There also exists exactly one idempotent e_i with $i \neq 1$ satisfying $Ue_i = 0$.

(iii) If $i = 2$, the only possible direct summand Y of U satisfying $Ye_1 \neq 0$ is $\begin{smallmatrix}1\\3\end{smallmatrix}$ so that $U = \begin{smallmatrix}1\\3\end{smallmatrix} \oplus V$ with $Ve_1 = 0$. Then, $P_1 \oplus U \in Q_n(\Lambda_n) \setminus \mathcal{S}$ is obvious and this is the only one possible case. In fact, by the analysis in (1) before, a completion of $\begin{smallmatrix}1\\3\end{smallmatrix} \oplus V$ to a τ-tilting Λ_n-module which does not belong to $\mathcal{S}$ is either of form $P_1 \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus W$ with $We_1 = We_2 = 0$ or of form $\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus 1 \oplus W$ with $We_1 = We_2 = We_3 = 0$, but the latter one is excluded if we restrict $Ve_1 = 0$. Therefore, $\mathfrak{q}$ in this subcase is defined by mapping U to $P_1 \oplus U$ and it is a bijection to the case (a2).

Similarly, if $i = 3$, we have $U = \begin{smallmatrix}1\\2\end{smallmatrix} \oplus V$ with $Ve_1 = 0$ and $P_1 \oplus U \in Q_n(\Lambda_n) \setminus \mathcal{S}$. Hence, $\mathfrak{q}$ is also defined by mapping U to $P_1 \oplus U$ and it is a bijection to the case (a1).

(iv) Suppose $i \geqslant 4$. Then, U has at least one of $\begin{smallmatrix}1\\2\end{smallmatrix}$, $\begin{smallmatrix}1\\3\end{smallmatrix}$, $\begin{smallmatrix}&1&\\2&&3\end{smallmatrix}$ and M_a as a direct summand.

- If U has exactly one of $\begin{smallmatrix}1\\2\end{smallmatrix}$ and $\begin{smallmatrix}1\\3\end{smallmatrix}$ as a direct summand, say, $U = \begin{smallmatrix}1\\2\end{smallmatrix} \oplus V$ with $Ve_1 = 0$, $Ue_3 \neq 0$ implies $S_3 \subseteq \mathsf{top}\ V$. Then, $\tau(\begin{smallmatrix}1\\2\end{smallmatrix}) = S_3$ indicates $\mathsf{Hom}_{\Lambda_n}(V, \tau(\begin{smallmatrix}1\\2\end{smallmatrix})) \neq 0$, contradicting with the assumption that U is a τ-rigid module. Also, one can get a contradiction for $U = \begin{smallmatrix}1\\3\end{smallmatrix} \oplus V$ with $Ve_1 = 0$.
- If $\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix}$ is a direct summand of U such that $U = \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus V$ with $Ve_1 = 0$, then $Ve_2 = Ve_3 = 0$ by the similar analysis with Case (b) in (1). This implies that V is a support τ-tilting $\Lambda_{[4,n]}$-module with support-rank $n-4$. However, $|U| = 2 + n - 4 = n - 2 < n - 1$, contradicting with $U \in Q_{n-1}(\Lambda_n)$.
- Otherwise, U must have one of $\begin{smallmatrix}&1&\\2&&3\end{smallmatrix}$ and M_a as a direct summand, so that $P_1 \oplus U \in Q_n(\Lambda_n) \setminus \mathcal{S}$ is well-defined by the analysis in (1). In this subcase, $\mathfrak{q}$ is defined by mapping U to $P_1 \oplus U$ and it is a bijection to (a3), (a4) and (a5).

We conclude that the map $\mathfrak{q}$ in this case is a bijection.

Lastly, it suffices to consider the following case.

(v) If $Ue_1 = 0$, $P_1 \oplus U \in Q_n(\Lambda_n) \setminus \mathcal{S}$ is well-defined. In fact, it follows from $\mathsf{Hom}_{\Lambda_n}(P_1, \tau U) = 0$. Thus, $\mathfrak{q}$ in this case is also defined by mapping U to $P_1 \oplus U$ and it is a bijection. This corresponds to the Case (a) in (1).

Now, we have found that $\mathfrak{q} : Q_{n-1}(\Lambda_n) \to Q_n(\Lambda_n) \setminus \mathcal{S}$ is a surjection, which is similar to the situation in Proposition A.2.4. In particular, the reason why $\mathfrak{q}$ is not an injection is explained in cases (i) and (ii). Then, the number of modules in $Q_n(\Lambda_n) \setminus \mathcal{S}$ is

$$a_{n-1}(\Lambda_n) - \#\left\{1 \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus V \mid V \in Q_{n-3}(\Lambda_{[4,n]})\right\} - \#\left\{1 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus Z \mid Z \in Q_{n-4}(\Lambda_{[4,n]})\right\}.$$

On the other hand, the number of modules in $\mathcal{S}$ is $a_{n-3}(\Lambda_{[4,n]}) = a_{n-3}(\mathbf{A}_{n-3})$. Hence, we conclude that

$$\begin{aligned} a_n(\Lambda_n) =& \#(Q_n(\Lambda_n) \setminus \mathcal{S}) + \#\mathcal{S} \\ =& a_{n-1}(\Lambda_n) - a_{n-3}(\mathbf{A}_{n-3}) - a_{n-4}(\mathbf{A}_{n-3}) + a_{n-3}(\mathbf{A}_{n-3}). \end{aligned}$$

Note that $a_{n-4}(\mathbf{A}_{n-3}) = a_{n-3}(\mathbf{A}_{n-3})$ by Lemma A.2.1. Then, the statement follows. $\square$

A.3 Supporting materials of Example 2.3.10

We recall the indecomposable projective A-modules P_i as follows,

$$P_1 = \begin{smallmatrix}1\\3\\2\end{smallmatrix}, P_2 = \begin{smallmatrix}2\\3\\1\\3\end{smallmatrix}, P_3 = \begin{smallmatrix}&3&\\&2&4\\1&&\\3&&\\2&&\end{smallmatrix}, \ P_4 = \begin{smallmatrix}4\\3\end{smallmatrix}.$$

Then, we construct three indecomposable A-modules to describe some basic τ-tilting modules. We first consider the τ-tilting A-module $P_1 \oplus P_2 \oplus P_3 \oplus P_4$ and take an exact sequence with a minimal left $\mathsf{add}(P_1 \oplus P_2 \oplus P_4)$-approximation π_1 of P_3:

$$P_3 \xrightarrow{\pi_1} P_1 \oplus P_2 \oplus P_4 \longrightarrow \mathsf{coker}\ \pi_1 \longrightarrow 0.$$

We define $M_1 := \mathsf{coker}\ \pi_1$ and $P_1 \oplus P_2 \oplus M_1 \oplus P_4$ is again a τ-tilting A-module. Then, we take an exact sequence with a minimal left $\mathsf{add}(P_2 \oplus M_1 \oplus P_4)$-approximation π_2 of P_1:

$$P_1 \xrightarrow{\pi_2} P_2 \oplus M_1 \longrightarrow \mathsf{coker}\ \pi_2 \longrightarrow 0$$

and define $M_2 := \mathsf{coker}\ \pi_2$. Last, we consider the τ-tilting A-module $\begin{smallmatrix}1\\3\\2\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\1&&4\\3&&\\2&&\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\\3\\2\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\1&&2\\3&&\\2&&\end{smallmatrix}$ (one may check this by Definition 2.1.1) and define $M_3 := \mathsf{coker}\ \pi_3$ as the cokernel of π_3, where π_3 is the minimal left $\mathsf{add}(\begin{smallmatrix}&3&\\1&&4\\3&&\\2&&\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\\3\\2\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\1&&2\\3&&\\2&&\end{smallmatrix})$-approximation with the following exact sequence:

$$\begin{smallmatrix}1\\3\\2\end{smallmatrix} \xrightarrow{\pi_3} \begin{smallmatrix}&3&\\1&&4\\3&&\\2&&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\1&&2\\3&&\\2&&\end{smallmatrix} \longrightarrow \mathsf{coker}\ \pi_3 \longrightarrow 0.$$

Next, by using $P_1, P_2, P_3, P_4, M_1, M_2, M_3$ and other explicitly described modules, we can give a complete list of τ-tilting $\widetilde{\mathcal{D}_4}$-modules by direct computation of left mutations.

$P_1 \oplus P_2 \oplus P_3 \oplus P_4$	$\begin{smallmatrix}&3&\\2&&4\end{smallmatrix} \oplus P_2 \oplus P_3 \oplus P_4$	$P_1 \oplus \begin{smallmatrix}&3&\\1&&4\\3&&\\2&&\end{smallmatrix} \oplus P_3 \oplus P_4$	$P_1 \oplus P_2 \oplus M_1 \oplus P_4$
$P_1 \oplus P_2 \oplus P_3 \oplus \begin{smallmatrix}&3&\\1&&2\\3&&\\2&&\end{smallmatrix}$	$\begin{smallmatrix}&3&\\2&&4\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\1&&4\\3&&\\2&&\end{smallmatrix} \oplus P_3 \oplus P_4$	$\begin{smallmatrix}&3&\\2&&4\end{smallmatrix} \oplus P_2 \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus P_4$	$\begin{smallmatrix}&3&\\2&&4\end{smallmatrix} \oplus P_2 \oplus P_3 \oplus \begin{smallmatrix}&3&\\1&&2\\3&&\\2&&\end{smallmatrix}$
$P_1 \oplus \begin{smallmatrix}&3&\\1&&4\\3&&\\2&&\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus P_4$	$P_1 \oplus P_2 \oplus M_1 \oplus \begin{smallmatrix}1&&2\\&3&\end{smallmatrix}$	$P_1 \oplus \begin{smallmatrix}1&&4\\&3&\end{smallmatrix} \oplus M_1 \oplus P_4$	$M_2 \oplus P_2 \oplus M_1 \oplus P_4$
$P_1 \oplus \begin{smallmatrix}&3&\\1&&4\\3&&\\2&&\end{smallmatrix} \oplus P_3 \oplus \begin{smallmatrix}&3&\\1&&2\\3&&\\2&&\end{smallmatrix}$	$\begin{smallmatrix}&3&\\2&&4\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\1&&4\\3&&\\2&&\end{smallmatrix} \oplus \begin{smallmatrix}3\\4\end{smallmatrix} \oplus P_4$	$M_2 \oplus P_2 \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus P_4$	$\begin{smallmatrix}&3&\\2&&4\end{smallmatrix} \oplus P_2 \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix}$
$\begin{smallmatrix}&3&\\2&&4\end{smallmatrix} \oplus P_2 \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\1&&2\\3&&\\2&&\end{smallmatrix}$	$\begin{smallmatrix}&3&\\1&&4\\3&&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\1&&4\\3&&\\2&&\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus P_4$	$P_1 \oplus \begin{smallmatrix}1&&4\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus P_4$	$P_1 \oplus \begin{smallmatrix}&3&\\1&&4\\3&&\\2&&\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\\3\\2\end{smallmatrix}$
$M_2 \oplus P_2 \oplus M_1 \oplus \begin{smallmatrix}1&&2\\&3&\end{smallmatrix}$	$P_1 \oplus \begin{smallmatrix}1&&4\\&3&\end{smallmatrix} \oplus M_1 \oplus \begin{smallmatrix}1&&2\\&3&\end{smallmatrix}$	$\begin{smallmatrix}2&&4\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}1&&4\\&3&\end{smallmatrix} \oplus M_1 \oplus P_4$	$M_2 \oplus \begin{smallmatrix}2&&4\\&3&\end{smallmatrix} \oplus M_1 \oplus P_4$
$P_1 \oplus \begin{smallmatrix}&3&\\1&&4\\3&&\\2&&\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\\3\\2\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\1&&2\\3&&\\2&&\end{smallmatrix}$	$\begin{smallmatrix}&3&\\2&&4\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\1&&4\\3&&\\2&&\end{smallmatrix} \oplus P_3 \oplus \begin{smallmatrix}&3&\\1&&2\\3&&\\2&&\end{smallmatrix}$	$\begin{smallmatrix}&3&\\1&&4\\3&&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\1&&4\\3&&\\2&&\end{smallmatrix} \oplus \begin{smallmatrix}3\\4\end{smallmatrix} \oplus P_4$	$M_2 \oplus \begin{smallmatrix}2&&4\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus P_4$
$M_2 \oplus P_2 \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}2&&\\3&&\\1&&2\\&3&\end{smallmatrix}$	$\begin{smallmatrix}&3&\\1&&4\\3&&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\1&&4\\3&&\\2&&\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\\3\\2\end{smallmatrix}$	$P_1 \oplus \begin{smallmatrix}1&&4\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus S_1$	$M_2 \oplus \begin{smallmatrix}2&&4\\&3&\end{smallmatrix} \oplus M_1 \oplus \begin{smallmatrix}1&&2\\&3&\end{smallmatrix}$
$M_2 \oplus P_2 \oplus \begin{smallmatrix}2&&\\3&&\\1&&2\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}1&&2\\&3&\end{smallmatrix}$	$\begin{smallmatrix}2&&4\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}1&&4\\&3&\end{smallmatrix} \oplus M_1 \oplus \begin{smallmatrix}1&&2\\&3&\end{smallmatrix}$	$P_1 \oplus \begin{smallmatrix}1&&4\\&3&\end{smallmatrix} \oplus S_1 \oplus \begin{smallmatrix}1&&2\\&3&\end{smallmatrix}$	$\begin{smallmatrix}2&&4\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}1&&4\\&3&\end{smallmatrix} \oplus S_4 \oplus P_4$
$M_3 \oplus \begin{smallmatrix}&3&\\1&&4\\3&&\\2&&\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\\3\\2\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\1&&2\\3&&\\2&&\end{smallmatrix}$	$\begin{smallmatrix}&3&\\2&&4\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\1&&4\\3&&\\2&&\end{smallmatrix} \oplus M_3 \oplus \begin{smallmatrix}&3&\\1&&2\\3&&\\2&&\end{smallmatrix}$	$\begin{smallmatrix}&3&\\1&&4\\3&&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\1&&4\\3&&\\2&&\end{smallmatrix} \oplus \begin{smallmatrix}3\\4\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\\3\\2\end{smallmatrix}$	$M_2 \oplus \begin{smallmatrix}2&&4\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}2&&\\3&&\\1&&2\\&3&\end{smallmatrix}$
$\begin{smallmatrix}&3&\\1&&4\\3&&\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\\3\\2\end{smallmatrix}$	$M_2 \oplus \begin{smallmatrix}2&&4\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}2&&\\3&&\\1&&2\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}1&&2\\&3&\end{smallmatrix}$	$\begin{smallmatrix}2&&4\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}1&&4\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}1&2&4\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}1&&2\\&3&\end{smallmatrix}$	$\begin{smallmatrix}1&2&4\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}1&&4\\&3&\end{smallmatrix} \oplus S_1 \oplus \begin{smallmatrix}1&&2\\&3&\end{smallmatrix}$
$\begin{smallmatrix}2&&4\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}1&&4\\&3&\end{smallmatrix} \oplus S_4 \oplus \begin{smallmatrix}1&2&4\\&3&\end{smallmatrix}$	$M_3 \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\\3\\2\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\1&&2\\3&&\\2&&\end{smallmatrix}$	$M_3 \oplus \begin{smallmatrix}&3&\\1&&4\\3&&\\2&&\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\\3\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\4\end{smallmatrix}$	$\begin{smallmatrix}&3&\\2&&4\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\1&&4\\3&&\\2&&\end{smallmatrix} \oplus M_3 \oplus \begin{smallmatrix}3\\4\end{smallmatrix}$
$\begin{smallmatrix}&3&\\2&&4\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus M_3 \oplus \begin{smallmatrix}&3&\\1&&2\\3&&\\2&&\end{smallmatrix}$	$\begin{smallmatrix}&3&\\1&&4\\3&&\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\4\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\\3\\2\end{smallmatrix}$	$S_2 \oplus \begin{smallmatrix}2&&4\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}2&&\\3&&\\1&&2\\&3&\end{smallmatrix}$	$S_2 \oplus \begin{smallmatrix}2&&4\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}2&&\\3&&\\1&&2\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}1&&2\\&3&\end{smallmatrix}$

$\begin{smallmatrix}2&&4\\&3&\end{smallmatrix} \oplus S_2 \oplus \begin{smallmatrix}1&2&4\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}1&&2\\&3&\end{smallmatrix}$	$\begin{smallmatrix}1&2&4\\&3&\end{smallmatrix} \oplus S_2 \oplus S_1 \oplus \begin{smallmatrix}1&&2\\&3&\end{smallmatrix}$	$\begin{smallmatrix}1&2&4\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}1&&4\\&3&\end{smallmatrix} \oplus S_1 \oplus S_4$	$\begin{smallmatrix}2&&4\\&3&\end{smallmatrix} \oplus S_2 \oplus S_4 \oplus \begin{smallmatrix}1&2&4\\&3&\end{smallmatrix}$
$M_3 \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\\3\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\4\end{smallmatrix}$	$\begin{smallmatrix}&3&\\2&&4\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus M_3 \oplus \begin{smallmatrix}3\\4\end{smallmatrix}$	$S_3 \oplus \begin{smallmatrix}3\\1\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\4\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\\3\\2\end{smallmatrix}$	$\begin{smallmatrix}1&2&4\\&3&\end{smallmatrix} \oplus S_2 \oplus S_1 \oplus S_4$
$S_3 \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\\3\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\4\end{smallmatrix}$			

A.4 Supporting materials of Lemma 5.2.3

We recall that the indecomposable projective $\widetilde{\mathcal{H}}_4$-modules are

$$P_1 = \begin{smallmatrix}1\\2\end{smallmatrix},\ P_2 = \begin{smallmatrix}&2&\\1&3&4\end{smallmatrix},\ P_3 = \begin{smallmatrix}3\\2\\4\end{smallmatrix}, P_4 = \begin{smallmatrix}4\\2\\3\end{smallmatrix}.$$

Then, we construct two indecomposable $\widetilde{\mathcal{H}}_4$-modules to describe some basic τ-tilting modules. We first consider the τ-tilting $\widetilde{\mathcal{H}}_4$-module $P_1 \oplus P_2 \oplus P_3 \oplus P_4$ and take an exact sequence with a minimal left $\mathsf{add}(P_1 \oplus P_3 \oplus P_4)$-approximation π_1 of P_2:

$$P_2 \xrightarrow{\pi_1} P_1 \oplus P_3 \oplus P_4 \longrightarrow \mathsf{coker}\ \pi_1 \longrightarrow 0.$$

We define $M_1 := \mathsf{coker}\ \pi_1$. Second, we consider the τ-tilting $\widetilde{\mathcal{H}}_4$-module $\begin{smallmatrix}&2&\\3&&4\end{smallmatrix} \oplus P_2 \oplus \begin{smallmatrix}&2&\\1&&4\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix}$ (one can check this by Definition 2.1.1) and define $M_2 := \mathsf{coker}\ \pi_2$ as the cokernel of π_2, where π_2 is the minimal left $\mathsf{add}(\begin{smallmatrix}&2&\\3&&4\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&4\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix})$-approximation with the exact sequence:

$$P_2 \xrightarrow{\pi_2} \begin{smallmatrix}&2&\\3&&4\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&4\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix} \longrightarrow \mathsf{coker}\ \pi_2 \longrightarrow 0.$$

Next, we can give a complete list of pairwise non-isomorphic basic τ-tilting $\widetilde{\mathcal{H}}_4$-modules by direct computation of left mutations, as displayed below.

$P_1 \oplus P_2 \oplus P_3 \oplus P_4$	$\begin{smallmatrix}&2&\\3&&4\end{smallmatrix} \oplus P_2 \oplus P_3 \oplus P_4$	$P_1 \oplus P_2 \oplus \begin{smallmatrix}&2&\\1&&4\end{smallmatrix} \oplus P_4$	$P_1 \oplus P_2 \oplus P_3 \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix}$
$P_1 \oplus M_1 \oplus P_3 \oplus P_4$	$\begin{smallmatrix}&2&\\3&&4\end{smallmatrix} \oplus P_2 \oplus \begin{smallmatrix}&2&\\1&&4\end{smallmatrix} \oplus P_4$	$\begin{smallmatrix}&2&\\3&&4\end{smallmatrix} \oplus P_2 \oplus P_3 \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix}$	$\begin{smallmatrix}3&&4\\&2&\end{smallmatrix} \oplus M_1 \oplus P_3 \oplus P_4$
$P_1 \oplus M_1 \oplus \begin{smallmatrix}1&&4\\&2&\end{smallmatrix} \oplus P_4$	$P_1 \oplus M_1 \oplus P_3 \oplus \begin{smallmatrix}1&&3\\&2&\end{smallmatrix}$	$P_1 \oplus \begin{smallmatrix}4\\2\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&4\end{smallmatrix} \oplus P_4$	$P_1 \oplus P_2 \oplus \begin{smallmatrix}&2&\\1&&4\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix}$
$P_1 \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus P_3 \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix}$	$\begin{smallmatrix}&2&\\3&&4\end{smallmatrix} \oplus \begin{smallmatrix}2\\4\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&4\end{smallmatrix} \oplus P_4$	$\begin{smallmatrix}&2&\\3&&4\end{smallmatrix} \oplus P_2 \oplus \begin{smallmatrix}&2&\\1&&4\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix}$	$\begin{smallmatrix}&2&\\3&&4\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus P_3 \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix}$
$\begin{smallmatrix}3&&4\\&2&\end{smallmatrix} \oplus M_1 \oplus \begin{smallmatrix}1&&4\\&2&\end{smallmatrix} \oplus P_4$	$P_1 \oplus M_1 \oplus \begin{smallmatrix}1&&4\\&2&\end{smallmatrix} \oplus \begin{smallmatrix}1&&3\\&2&\end{smallmatrix}$	$\begin{smallmatrix}3&&4\\&2&\end{smallmatrix} \oplus M_1 \oplus P_3 \oplus \begin{smallmatrix}1&&3\\&2&\end{smallmatrix}$	$\begin{smallmatrix}2\\4\end{smallmatrix} \oplus \begin{smallmatrix}4\\2\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&4\end{smallmatrix} \oplus P_4$
$P_1 \oplus \begin{smallmatrix}4\\2\end{smallmatrix} \oplus \begin{smallmatrix}1&&4\\&2&\end{smallmatrix} \oplus P_4$	$P_1 \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&4\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix}$	$P_1 \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus P_3 \oplus \begin{smallmatrix}1&&3\\&2&\end{smallmatrix}$	$\begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus P_3 \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix}$
$\begin{smallmatrix}&2&\\3&&4\end{smallmatrix} \oplus \begin{smallmatrix}2\\4\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&4\end{smallmatrix} \oplus M_2$	$\begin{smallmatrix}&2&\\3&&4\end{smallmatrix} \oplus M_2 \oplus \begin{smallmatrix}&2&\\1&&4\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix}$	$\begin{smallmatrix}&2&\\3&&4\end{smallmatrix} \oplus M_2 \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix}$	$\begin{smallmatrix}3&&4\\&2&\end{smallmatrix} \oplus M_1 \oplus \begin{smallmatrix}1&&4\\&2&\end{smallmatrix} \oplus \begin{smallmatrix}1&&3\\&2&\end{smallmatrix}$
$\begin{smallmatrix}3&&4\\&2&\end{smallmatrix} \oplus S_4 \oplus \begin{smallmatrix}1&&4\\&2&\end{smallmatrix} \oplus P_4$	$P_1 \oplus S_1 \oplus \begin{smallmatrix}1&&4\\&2&\end{smallmatrix} \oplus \begin{smallmatrix}1&&3\\&2&\end{smallmatrix}$	$\begin{smallmatrix}3&&4\\&2&\end{smallmatrix} \oplus S_3 \oplus P_3 \oplus \begin{smallmatrix}1&&3\\&2&\end{smallmatrix}$	$\begin{smallmatrix}4\\2\end{smallmatrix} \oplus S_4 \oplus \begin{smallmatrix}1&&4\\&2&\end{smallmatrix} \oplus P_4$

$M_2 \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&4\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix}$	$\begin{smallmatrix}3\\2\end{smallmatrix} \oplus S_3 \oplus P_3 \oplus \begin{smallmatrix}1&&3\\&2&\end{smallmatrix}$	$\begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\4\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&4\end{smallmatrix} \oplus M_2$	$\begin{smallmatrix}&2&\\3&&4\end{smallmatrix} \oplus M_2 \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\4\end{smallmatrix}$
$\begin{smallmatrix}3&&4\\&2&\end{smallmatrix} \oplus \begin{smallmatrix}1\,3\,4\\2\end{smallmatrix} \oplus \begin{smallmatrix}1&&4\\&2&\end{smallmatrix} \oplus \begin{smallmatrix}1&&3\\&2&\end{smallmatrix}$	$\begin{smallmatrix}3&&4\\&2&\end{smallmatrix} \oplus S_4 \oplus \begin{smallmatrix}1&&4\\&2&\end{smallmatrix} \oplus \begin{smallmatrix}1\,3\,4\\2\end{smallmatrix}$	$\begin{smallmatrix}1\,3\,4\\2\end{smallmatrix} \oplus S_1 \oplus \begin{smallmatrix}1&&4\\&2&\end{smallmatrix} \oplus \begin{smallmatrix}1&&3\\&2&\end{smallmatrix}$	$\begin{smallmatrix}3&&4\\&2&\end{smallmatrix} \oplus S_3 \oplus \begin{smallmatrix}1\,3\,4\\2\end{smallmatrix} \oplus \begin{smallmatrix}1&&3\\&2&\end{smallmatrix}$
$M_2 \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix}$	$M_2 \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\4\end{smallmatrix}$	$S_2 \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\4\end{smallmatrix}$	$S_1 \oplus S_4 \oplus \begin{smallmatrix}1&&4\\&2&\end{smallmatrix} \oplus \begin{smallmatrix}1\,3\,4\\2\end{smallmatrix}$
$\begin{smallmatrix}3&&4\\&2&\end{smallmatrix} \oplus S_4 \oplus S_3 \oplus \begin{smallmatrix}1\,3\,4\\2\end{smallmatrix}$	$S_1 \oplus S_3 \oplus \begin{smallmatrix}1\,3\,4\\2\end{smallmatrix} \oplus \begin{smallmatrix}1&&3\\&2&\end{smallmatrix}$	$S_1 \oplus S_4 \oplus S_3 \oplus \begin{smallmatrix}1\,3\,4\\2\end{smallmatrix}$	

A.5 Supporting materials of Proposition 5.3.1

We recall that the indecomposable projective $\widetilde{\mathcal{K}_4}$-modules P_1, P_2, P_4 are

$$P_1 = \begin{smallmatrix}1\\2\\3\end{smallmatrix},\ P_2 = \begin{smallmatrix}&2&\\1&&3\\&&4\end{smallmatrix},\ P_4 = \begin{smallmatrix}4\\3\\2\\3\end{smallmatrix}.$$

We may look at the structures of the indecomposable projective $\widetilde{\mathcal{K}_4}$-module P_3 and the indecomposable injective $\widetilde{\mathcal{K}_4}$-module I_3 in detail, that is,

$$P_3 = \begin{smallmatrix}&&3&\\&2&&4\\1&&3&\\&&4&\end{smallmatrix} \text{ and } I_3 = \begin{smallmatrix}&&4&\\1&&3&\\&2&&4\\&&3&\end{smallmatrix}.$$

Then, we construct two indecomposable $\widetilde{\mathcal{K}_4}$-modules to describe some basic τ-tilting modules. We first consider the τ-tilting $\widetilde{\mathcal{K}_4}$-module $P_1 \oplus P_2 \oplus \begin{smallmatrix}&2&&4\\1&&3&\end{smallmatrix} \oplus P_4$ and take an exact sequence with a minimal left $\mathsf{add}(P_1 \oplus \begin{smallmatrix}&2&&4\\1&&3&\end{smallmatrix} \oplus P_4)$-approximation π_1 of P_2:

$$P_2 \xrightarrow{\pi_1} P_1 \oplus \begin{smallmatrix}&2&&4\\1&&3&\end{smallmatrix} \oplus P_4 \longrightarrow \mathsf{coker}\ \pi_1 \longrightarrow 0.$$

We define $M_1 := \mathsf{coker}\ \pi_1$. Second, we consider the τ-tilting $\widetilde{\mathcal{K}_4}$-module $P_1 \oplus \begin{smallmatrix}&3&\\&2&\\1&&3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&2&\\1&&3\\&&4\end{smallmatrix}$ (one can check this by Definition 2.1.1) and define $M_2 := \mathsf{coker}\ \pi_2$ as the cokernel of π_2, where π_2 is the minimal left $\mathsf{add}(\begin{smallmatrix}&3&\\&2&\\1&&3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&2&\\1&&3\\&&4\end{smallmatrix})$-approximation with the exact sequence:

$$P_1 \xrightarrow{\pi_2} \begin{smallmatrix}&2&\\1&&3\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&2&\\1&&3\\&&4\end{smallmatrix} \longrightarrow \mathsf{coker}\ \pi_2 \longrightarrow 0.$$

Similarly, by using P_1, P_2, P_3, P_4, M_1, M_2 and other explicitly described modules, we can give a complete list of pairwise non-isomorphic basic τ-tilting $\widetilde{\mathcal{K}_4}$-modules by direct computation of left mutations.

$P_1 \oplus P_2 \oplus P_3 \oplus P_4$	$\begin{smallmatrix}2\\3\\4\end{smallmatrix} \oplus P_2 \oplus P_3 \oplus P_4$	$P_1 \oplus P_2 \oplus \begin{smallmatrix}&2&&4\\1&&3&\end{smallmatrix} \oplus P_4$	$P_1 \oplus P_2 \oplus P_3 \oplus \begin{smallmatrix}&3&\\&2&\\1&&3\\&&4\end{smallmatrix}$
$P_1 \oplus \begin{smallmatrix}1&&3&\\&2&&4\\&3&&\end{smallmatrix} \oplus P_3 \oplus P_4$	$\begin{smallmatrix}2\\3\\4\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\2&&4\\3&&\\4&&\end{smallmatrix} \oplus P_3 \oplus P_4$	$\begin{smallmatrix}2\\3\\4\end{smallmatrix} \oplus P_2 \oplus \begin{smallmatrix}&2&&4\\1&&3&\end{smallmatrix} \oplus P_4$	$\begin{smallmatrix}2\\3\\4\end{smallmatrix} \oplus P_2 \oplus P_3 \oplus \begin{smallmatrix}&3&\\&2&\\1&&3\\&&4\end{smallmatrix}$
$\begin{smallmatrix}3\\4\end{smallmatrix} \oplus \begin{smallmatrix}1&&3&\\&2&&4\\&3&&\end{smallmatrix} \oplus P_3 \oplus P_4$	$P_1 \oplus \begin{smallmatrix}1&&3&\\&2&&4\\&3&&\end{smallmatrix} \oplus \begin{smallmatrix}&&4\\1&&3\\&2&\\&3&\end{smallmatrix} \oplus P_4$	$P_1 \oplus \begin{smallmatrix}1&&3&\\&2&&4\\&3&&\end{smallmatrix} \oplus P_3 \oplus \begin{smallmatrix}&3&\\&2&\\1&&3\\&&4\end{smallmatrix}$	$P_1 \oplus M_1 \oplus \begin{smallmatrix}&2&&4\\1&&3&\end{smallmatrix} \oplus P_4$

$P_1 \oplus P_2 \oplus \begin{smallmatrix}&2&&4\\1&&3&\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix}$	$P_1 \oplus P_2 \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&2&\\1&&3\\&&4\end{smallmatrix}$	$\begin{smallmatrix}3\\4\end{smallmatrix} \oplus \begin{smallmatrix}3&\\2&4\\3&\\4&\end{smallmatrix} \oplus P_3 \oplus P_4$	$\begin{smallmatrix}2\\3\\4\end{smallmatrix} \oplus \begin{smallmatrix}3&\\2&4\\3&\\4&\end{smallmatrix} \oplus P_3 \oplus \begin{smallmatrix}&3&\\&2&\\1&&3\\&&4\end{smallmatrix}$
$\begin{smallmatrix}2\\3\\4\end{smallmatrix} \oplus \begin{smallmatrix}2&&4\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}&2&&4\\1&&3&\end{smallmatrix} \oplus P_4$	$\begin{smallmatrix}2\\3\\4\end{smallmatrix} \oplus P_2 \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&2&\\1&&3\\&&4\end{smallmatrix}$	$\begin{smallmatrix}3\\4\end{smallmatrix} \oplus \begin{smallmatrix}1&&3&\\&2&&4\\&3&&\end{smallmatrix} \oplus \begin{smallmatrix}&&4\\1&&3\\&2&\\&3&\end{smallmatrix} \oplus P_4$	$P_1 \oplus \begin{smallmatrix}1&&3&\\&2&&4\\&3&&\end{smallmatrix} \oplus \begin{smallmatrix}&&4\\1&&3\\&2&\\&3&\end{smallmatrix} \oplus S_1$
$\begin{smallmatrix}3\\4\end{smallmatrix} \oplus \begin{smallmatrix}1&&3&\\&2&&4\\&3&&\end{smallmatrix} \oplus P_3 \oplus \begin{smallmatrix}&3&\\&2&\\1&&3\\&&4\end{smallmatrix}$	$P_1 \oplus \begin{smallmatrix}1&&3&\\&2&&4\\&3&&\end{smallmatrix} \oplus \begin{smallmatrix}1&&3\\&2&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&2&\\1&&3\\&&4\end{smallmatrix}$	$P_1 \oplus M_1 \oplus \begin{smallmatrix}&&4\\1&&3\\&2&\\&3&\end{smallmatrix} \oplus P_4$	$\begin{smallmatrix}4&\\3&\\2&4\\3&\end{smallmatrix} \oplus M_1 \oplus \begin{smallmatrix}&2&&4\\1&&3&\end{smallmatrix} \oplus P_4$
$P_1 \oplus M_1 \oplus \begin{smallmatrix}&2&&4\\1&&3&\end{smallmatrix} \oplus \begin{smallmatrix}1&\\2&4\\3&\end{smallmatrix}$	$\begin{smallmatrix}2\\3\\4\end{smallmatrix} \oplus P_2 \oplus \begin{smallmatrix}&2&&4\\1&&3&\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix}$	$P_1 \oplus \begin{smallmatrix}1&\\2&4\\3&\end{smallmatrix} \oplus \begin{smallmatrix}&2&&4\\1&&3&\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix}$	$P_1 \oplus P_2 \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix}$
$P_1 \oplus \begin{smallmatrix}&3&\\&2&\\1&&3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&2&\\1&&3\\&&4\end{smallmatrix}$	$\begin{smallmatrix}3\\4\end{smallmatrix} \oplus \begin{smallmatrix}3&\\2&4\\3&\\4&\end{smallmatrix} \oplus P_3 \oplus \begin{smallmatrix}&3&\\&2&\\1&&3\\&&4\end{smallmatrix}$	$\begin{smallmatrix}2\\3\\4\end{smallmatrix} \oplus \begin{smallmatrix}3&\\2&4\\3&\\4&\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\\3\\4\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&2&\\1&&3\\&&4\end{smallmatrix}$	$\begin{smallmatrix}4&\\3&\\2&4\\3&\end{smallmatrix} \oplus \begin{smallmatrix}2&&4\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}&2&&4\\1&&3&\end{smallmatrix} \oplus P_4$
$\begin{smallmatrix}2\\3\\4\end{smallmatrix} \oplus \begin{smallmatrix}2&&4\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}&2&&4\\1&&3&\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix}$	$\begin{smallmatrix}2\\3\\4\end{smallmatrix} \oplus M_2 \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&2&\\1&&3\\&&4\end{smallmatrix}$	$\begin{smallmatrix}3\\4\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}&&4\\1&&3\\&2&\\&3&\end{smallmatrix} \oplus P_4$	$\begin{smallmatrix}3\\4\end{smallmatrix} \oplus \begin{smallmatrix}1&&3&\\&2&&4\\&3&&\end{smallmatrix} \oplus \begin{smallmatrix}&&4\\1&&3\\&2&\\&3&\end{smallmatrix} \oplus S_1$
$P_1 \oplus \begin{smallmatrix}1&\\2&4\\3&\end{smallmatrix} \oplus \begin{smallmatrix}&&4\\1&&3\\&2&\\&3&\end{smallmatrix} \oplus S_1$	$P_1 \oplus \begin{smallmatrix}1&&3&\\&2&&4\\&3&&\end{smallmatrix} \oplus \begin{smallmatrix}1&&3\\&2&\\&3&\end{smallmatrix} \oplus S_1$	$\begin{smallmatrix}3\\4\end{smallmatrix} \oplus \begin{smallmatrix}1&&3&\\&2&&4\\&3&&\end{smallmatrix} \oplus \begin{smallmatrix}1&&3\\&2&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&2&\\1&&3\\&&4\end{smallmatrix}$	$\begin{smallmatrix}4&\\3&\\2&4\\3&\end{smallmatrix} \oplus M_1 \oplus \begin{smallmatrix}&&4\\1&&3\\&2&\\&3&\end{smallmatrix} \oplus P_4$
$P_1 \oplus M_1 \oplus \begin{smallmatrix}&&4\\1&&3\\&2&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}1&\\2&4\\3&\end{smallmatrix}$	$\begin{smallmatrix}4&\\3&\\2&4\\3&\end{smallmatrix} \oplus M_1 \oplus \begin{smallmatrix}&2&&4\\1&&3&\end{smallmatrix} \oplus \begin{smallmatrix}1&\\2&4\\3&\end{smallmatrix}$	$\begin{smallmatrix}2\\3\\4\end{smallmatrix} \oplus P_2 \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix}$	$S_4 \oplus \begin{smallmatrix}1&\\2&4\\3&\end{smallmatrix} \oplus \begin{smallmatrix}&2&&4\\1&&3&\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix}$
$P_1 \oplus \begin{smallmatrix}1&\\2&4\\3&\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix}$	$M_2 \oplus \begin{smallmatrix}&3&\\&2&\\1&&3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&2&\\1&&3\\&&4\end{smallmatrix}$	$P_1 \oplus \begin{smallmatrix}&3&\\&2&\\1&&3\end{smallmatrix} \oplus \begin{smallmatrix}1&&3\\&2&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&2&\\1&&3\\&&4\end{smallmatrix}$	$\begin{smallmatrix}3\\4\end{smallmatrix} \oplus \begin{smallmatrix}3&\\2&4\\3&\\4&\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\\3\\4\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&2&\\1&&3\\&&4\end{smallmatrix}$
$\begin{smallmatrix}3\\4\end{smallmatrix} \oplus S_3 \oplus \begin{smallmatrix}3\\2\\3\\4\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&2&\\1&&3\\&&4\end{smallmatrix}$	$\begin{smallmatrix}4&\\3&\\2&4\\3&\end{smallmatrix} \oplus \begin{smallmatrix}2&&4\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}&2&&4\\1&&3&\end{smallmatrix} \oplus S_4$	$S_4 \oplus \begin{smallmatrix}2&&4\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}&2&&4\\1&&3&\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix}$	$\begin{smallmatrix}2\\3\\4\end{smallmatrix} \oplus \begin{smallmatrix}2&&4\\&3&\end{smallmatrix} \oplus S_2 \oplus \begin{smallmatrix}2\\1\end{smallmatrix}$
$S_4 \oplus \begin{smallmatrix}2&&4\\&3&\end{smallmatrix} \oplus S_2 \oplus \begin{smallmatrix}2\\1\end{smallmatrix}$	$\begin{smallmatrix}2\\3\\4\end{smallmatrix} \oplus M_2 \oplus \begin{smallmatrix}3\\2\\3\\4\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&2&\\1&&3\\&&4\end{smallmatrix}$	$\begin{smallmatrix}2\\3\\4\end{smallmatrix} \oplus M_2 \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix}$	$\begin{smallmatrix}2\\3\\4\end{smallmatrix} \oplus M_2 \oplus \begin{smallmatrix}3\\2\\3\\4\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix}$
$\begin{smallmatrix}3\\4\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}&&4\\1&&3\\&2&\\&3&\end{smallmatrix} \oplus S_1$	$\begin{smallmatrix}3\\4\end{smallmatrix} \oplus \begin{smallmatrix}1&&3&\\&2&&4\\&3&&\end{smallmatrix} \oplus \begin{smallmatrix}1&&3\\&2&\\&3&\end{smallmatrix} \oplus S_1$	$\begin{smallmatrix}3\\4\end{smallmatrix} \oplus S_3 \oplus \begin{smallmatrix}1&&3\\&2&\\&3&\end{smallmatrix} \oplus S_1$	$I_3 \oplus \begin{smallmatrix}1&\\2&4\\3&\end{smallmatrix} \oplus \begin{smallmatrix}&&4\\1&&3\\&2&\\&3&\end{smallmatrix} \oplus S_1$
$P_1 \oplus \begin{smallmatrix}1&\\2&4\\3&\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus S_1$	$I_3 \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}&&4\\1&&3\\&2&\\&3&\end{smallmatrix} \oplus S_1$	$I_3 \oplus \begin{smallmatrix}1&\\2&4\\3&\end{smallmatrix} \oplus S_4 \oplus S_1$	$I_3 \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus S_4 \oplus S_1$
$\begin{smallmatrix}3\\4\end{smallmatrix} \oplus S_3 \oplus \begin{smallmatrix}1&&3\\&2&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&2&\\1&&3\\&&4\end{smallmatrix}$	$\begin{smallmatrix}4&\\3&\\2&4\\3&\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}&&4\\1&&3\\&2&\\&3&\end{smallmatrix} \oplus P_4$	$\begin{smallmatrix}4&\\3&\\2&4\\3&\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}&&4\\1&&3\\&2&\\&3&\end{smallmatrix} \oplus I_3$	$\begin{smallmatrix}4&\\3&\\2&4\\3&\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus S_4 \oplus I_3$
$\begin{smallmatrix}4&\\3&\\2&4\\3&\end{smallmatrix} \oplus M_1 \oplus \begin{smallmatrix}&&4\\1&&3\\&2&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}1&\\2&4\\3&\end{smallmatrix}$	$\begin{smallmatrix}4&\\3&\\2&4\\3&\end{smallmatrix} \oplus I_3 \oplus \begin{smallmatrix}&&4\\1&&3\\&2&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}1&\\2&4\\3&\end{smallmatrix}$	$\begin{smallmatrix}4&\\3&\\2&4\\3&\end{smallmatrix} \oplus S_4 \oplus \begin{smallmatrix}&2&&4\\1&&3&\end{smallmatrix} \oplus \begin{smallmatrix}1&\\2&4\\3&\end{smallmatrix}$	$\begin{smallmatrix}4&\\3&\\2&4\\3&\end{smallmatrix} \oplus S_4 \oplus I_3 \oplus \begin{smallmatrix}1&\\2&4\\3&\end{smallmatrix}$
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